11

Integral Calculus.

Partial Differentiation

(Do All examples)
(Pf of all theorems in main text do)

CONTENTS

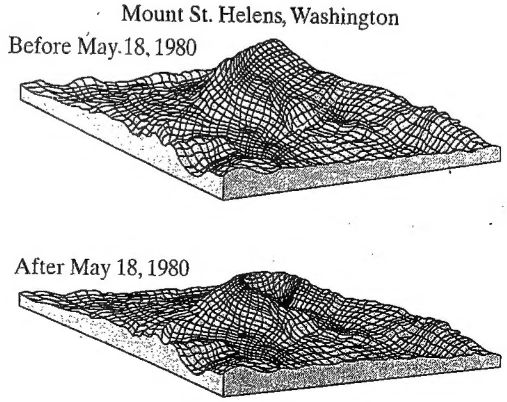
PREVIEW

This chapter extends the methods of single-variable differential calculus to functions of two or more independent variables. You will learn how to take derivatives of such functions and to interpret those derivatives as slopes and rates of change. You will also learn a more general version of the chain rule and extended procedures for optimizing a function. The vector methods developed in Chapters 9 and 10 play an important role in this chapter, and indeed, we will find that the closest analogue of the single-variable derivative is a certain vector function called the *gradient*. In physics, the gradient is the rate at which a variable quantity, such as temperature or pressure, changes in value. In this chapter, we will define the *gradient of a function*, which is a vector whose components along each axis are related to the rate at which the function changes in the direction of that axis.

PERSPECTIVE

In many practical situations, the value of one quantity depends on the values of two or more others. For example, the amount of water in a reservoir depends on the amount of rainfall and on the amount of water consumed by local residents and thus may be regarded as a function of two independent variables. The current in an electrical circuit is a function of four variables: the electromotive force, the capacitance, the resistance, and the inductance. Graphical models involving functions of several variables often provide a valuable picture of what is happening in a situation of interest, as illustrated by the graphical model of Mount St. Helens shown in the accompanying figure before and after its eruption on May 18, 1980. We will analyze a variety of such models using techniques and tools that will be developed in this chapter.

Mount St. Helens, Washington



Mount St. Helens map courtesy of Bill Lennox, Humboldt State University.

11.1 Functions of Several Variables

IN THIS SECTION basic concepts, level curves and surfaces, graphs of functions of two variables

BASIC CONCEPTS

Physical quantities often depend on two or more variables (see the Perspective at the beginning of this chapter). For example, we might be concerned with the temperature, T, at various points (x, y) on a metal plate. In this situation, T might be regarded as a function of the two location variables, x and y. Extending the notation for a function of a single variable, we denote this relationship by $T(x, y)$.

Function of Two Variables

> A **function of two variables** is a rule f that assigns to each ordered pair (x, y) in a set D a unique number $f(x, y)$. The set D is called the **domain** of the function, and the corresponding values of $f(x, y)$ constitute the **range** of f.

Functions of three or more variables can be defined in a similar fashion. For example, the temperature in our introductory example may vary not only on the plate, but also with time t, in which case it would be denoted by $T(x, y, t)$. Occasionally, we will examine functions of four or more variables, but for simplicity, we will focus most of our attention on functions of two or three variables.

When dealing with a function f of two variables, we may write $z = f(x, y)$ and refer to x and y as the **independent variables** and to z as the **dependent variable**. Often, the functional "rule" will be given as a formula, and unless otherwise stated, *we will assume that the domain is the largest set of points in the plane (or in* $\mathbb{R}^3$*) for which the functional formula is defined and real-valued.* These definitions and conventions are illustrated in Example 1 for a function of two variables.

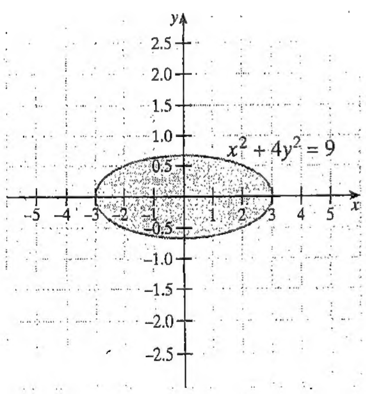


Figure 11.1 The domain of $f(x, y) = \sqrt{9 - x^2 - 4y^2}$

EXAMPLE 1 Domain, range, and evaluating a function of two variables

Let $f(x, y) = \sqrt{9 - x^2 - 4y^2}$.

a. Evaluate $f(2, 1)$ and $f(2t, t^2)$.

b. Describe the domain and range of f.

Solution

a. $f(2, 1) = \sqrt{9 - 2^2 - 4(1)^2} = 1$

$f(2t, t^2) = \sqrt{9 - (2t)^2 - 4(t^2)^2} = \sqrt{9 - 4t^2 - 4t^4}$

b. The domain of f is the set of all ordered pairs (x, y) for which $\sqrt{9 - x^2 - 4y^2}$ is defined. We must have $9 - x^2 - 4y^2 \geq 0$, or, equivalently, $x^2 + 4y^2 \leq 9$, in order for the square root to be defined. Thus, the domain of f is the set of all points (x, y) inside or on the ellipse $x^2 + 4y^2 = 9$, as shown in Figure 11.1.

The range of f is the set of all numbers $z = \sqrt{9 - x^2 - 4y^2}$ for (x, y) in the domain $x^2 + 4y^2 \leq 9$. Thus, the range is the interval $0 \leq z \leq 3$. ■

Functions of several variables can be combined in much the same way as functions of a single variable, as noted in the following definition.

Operations with Functions of Two Variables

If $f(x, y)$ and $g(x, y)$ are functions of two variables with domain D, then

Sum	$(f+g)(x, y) = f(x, y) + g(x, y)$
Difference	$(f-g)(x, y) = f(x, y) - g(x, y)$
Product	$(fg)(x, y) = f(x, y)g(x, y)$
Quotient	$\left(\dfrac{f}{g}\right)(x, y) = \dfrac{f(x, y)}{g(x, y)} \qquad g(x, y) \neq 0$

A **polynomial function in x and y** is a sum of functions of the form

$$Cx^m y^n$$

with nonnegative integers m and n and C a constant; for instance,

$$3x^5y^3 - 7x^2y + 2x - 3y + 11$$

is a polynomial in x and y. A **rational function** is a quotient of two polynomial functions. Similar notation and terminology apply to functions of three or more variables.

LEVEL CURVES AND SURFACES

By analogy with the single-variable case, we define the **graph of the function** $f(x, y)$ to be the collection of all 3-tuples (ordered triples) (x, y, z) such that (x, y) is in the domain of f and $z = f(x, y)$. The graph of $f(x, y)$ is a surface in $\mathbb{R}^3$ whose projection onto the xy-plane is the domain D.

It is usually not easy to sketch the graph of a function of two variables without the assistance of technology. One way to proceed is illustrated in Figure 11.2.*

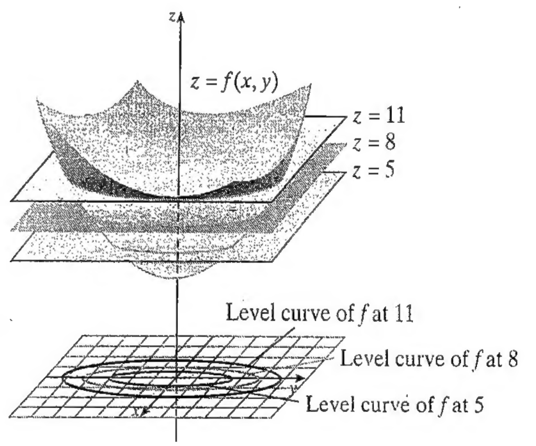


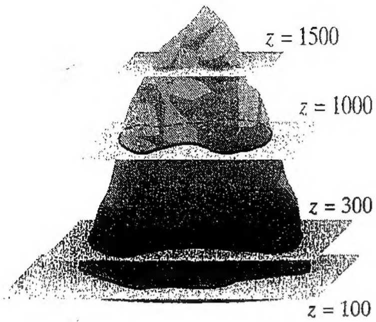
Figure 11.2 Graph of a function of two variables

Notice that when the plane $z = C$ intersects the surface $z = f(x, y)$, the result is the curve with the equation $f(x, y) = C$. Such an intersection is called the **trace** of the graph of f in the plane $z = C$. The set of points (x, y) in the xy-plane that satisfy $f(x, y) = C$ is called the **level curve** of f at C, and an entire family of level curves is generated as C varies over the range of f. We can think of a trace as a "slice" of the surface at a particular location and a level curve as its projection onto the xy-plane. By sketching members of this family on the xy-plane, we obtain a useful topographical map of the surface $z = f(x, y)$. Because level curves are used to show the shape of a surface (a mountain, for example), they are sometimes called **contour curves**.

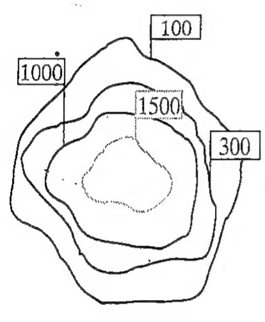
For instance, imagine that the surface $z = f(x, y)$ is a "mountain" and that we wish to draw a two-dimensional "profile" of its shape. To draw such a profile, we indicate

*See the *Technology Manual* accompanying this book for some suggestions about using technology to help you graph functions of two variables.

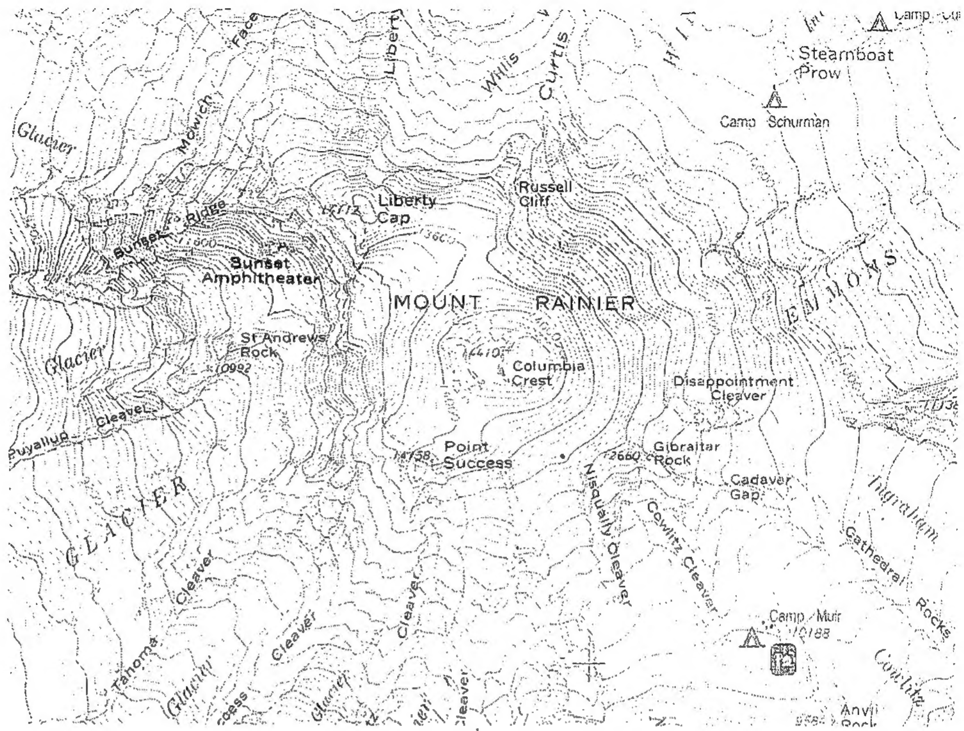
the paths of constant elevation by sketching the family of level curves in the plane and pinning a "flag" to each curve to show the elevation to which it corresponds, as shown in Figure 11.3b. Notice that regions in the map where paths are crowded together correspond to the steeper portions of the mountain. An actual topographical map of Mount Rainier is shown in Figure 11.3c.



a. The surface $z = f(x, y)$ as a mountain



b. Level curves yield a topographic map of $z = f(x, y)$



c. Topographic map of Mount Rainier

Figure 11.3 Level curves of a surface

You probably have seen level curves on the weather report in the newspaper or television news shows, where level curves of equal temperature are called **isotherms** (see Figure 11.4). Other common uses of level curves involve curves of equal pressure (called **isobars**) and curves of electric potential (called **equipotential lines**).

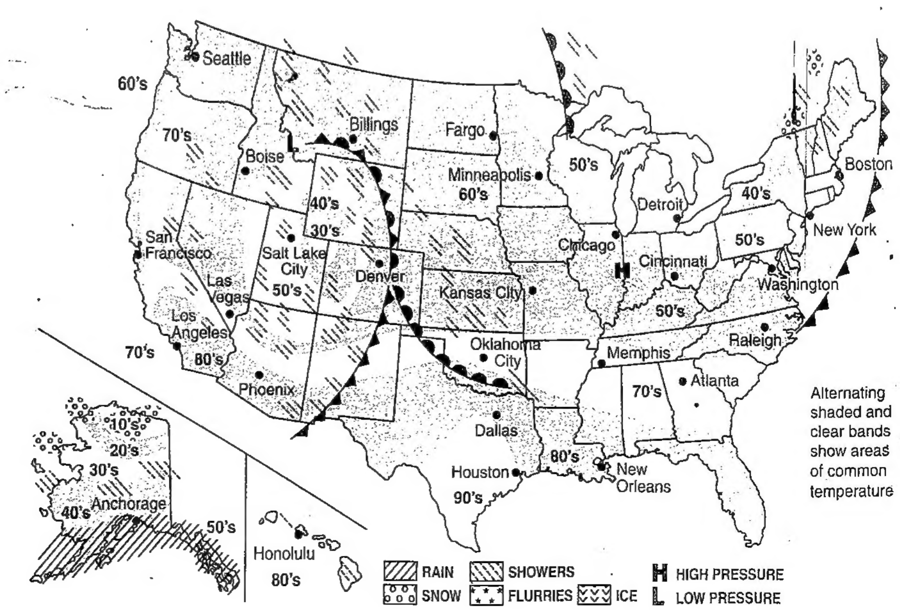


Figure 11.4 Isotherms

EXAMPLE 2 Level curves

Sketch some level curves of the function $f(x, y) = 10 - x^2 - y^2$.

Solution

A computer graph of $z = f(x, y)$ is the surface shown in Figure 11.5a. Figure 11.5b shows traces of the graph of f in the planes $z = 1$, $z = 6$, and $z = 9$, and the corresponding level curves are shown in Figure 11.5c. A table of values is shown in the margin.

k	level curve $z = k$
1	$1 = 10 - x^2 - y^2$ or $x^2 + y^2 = 9$
6	$6 = 10 - x^2 - y^2$ or $x^2 + y^2 = 4$
9	$9 = 10 - x^2 - y^2$ or $x^2 + y^2 = 1$

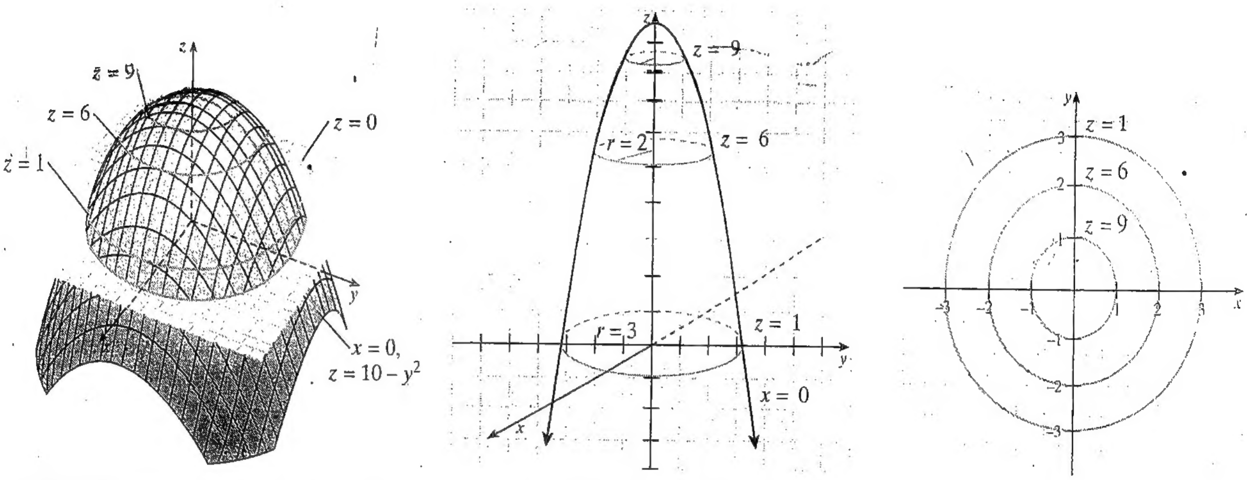


Figure 11.5 Three ways of viewing the surface $z = 10 - x^2 - y^2$

GRAPHS OF FUNCTIONS OF TWO VARIABLES

The level curves of a function $f(x, y)$ provide information about the cross sections of the surface $z = f(x, y)$ perpendicular to the z-axis. However, a more complete picture of the surface can often be obtained by examining cross sections in other directions as well. This procedure is used to graph a function in Example 3.

EXAMPLE 3 Level curves

Use the level curves of the function $f(x, y) = x^2 + y^2$ to sketch the graph of f.

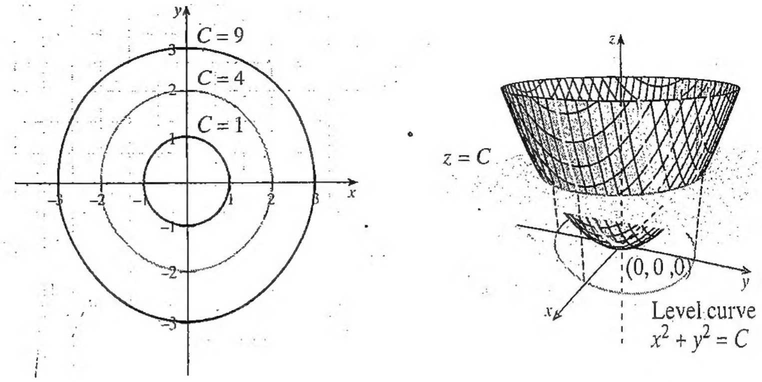
Solution

The level curve $x^2 + y^2 = 0$ (that is, $C = 0$) is the point $(0, 0)$, and for $C > 0$, the level curve $x^2 + y^2 = C$ is the circle with center $(0, 0)$ and radius $\sqrt{C}$ (Figure 11.6a). There are no points (x, y) that satisfy $x^2 + y^2 = C$ for $C < 0$.

We can gain additional information about the appearance of the surface by examining cross sections perpendicular to the other two principal directions; that is, to the x-axis and the y-axis. Cross-sectional planes perpendicular to the x-axis have the form $x = A$ and intersect the surface $z = x^2 + y^2$ in parabolas of the form $z = A^2 + y^2$. For example, the cross section of the plane $x = 5$ intersects the surface in the parabola $z = 25 + y^2$. That is, it is the set of point $(5, y, 25 + y^2)$ as y varies. Similarly, cross-sectional planes perpendicular to the y-axis have the form $y = B$ and intersect the surface in parabolas of the form $z = x^2 + B^2$.

To summarize, the surface $z = x^2 + y^2$ has cross sections that are circles in

Since the surface is formed by revolving a parabola about its axis, it is called a **circular paraboloid** or a **paraboloid of revolution**. The graph of the surface is shown in Figure 11.6b.



a. Level curves of $f(x, y) = x^2 + y^2$ are circles in the form $x^2 + y^2 = C$.

b. The surface $z = x^2 + y^2$. Cross sections perpendicular to the y-axis shown in white are parabolas.

Figure 11.6 The graph of the function $f(x, y) = x^2 + y^2$ ■

The concept of level curve can be generalized to apply to functions of more than two variables. In particular, if f is a function of three variables x, y, and z, then the graph of the equation $f(x, y, z) = C$ is a surface of $\mathbb{R}^3$ called a **level surface** of f at C.

EXAMPLE 4 Isothermal surface

Suppose a region of $\mathbb{R}^3$ is heated so that its temperature T at each point (x, y, z) is given by $T(x, y, z) = 100 - x^2 - y^2 - z^2$ degrees Celsius. Describe the isothermal surfaces for $T > 0$.

Solution

The isothermal surfaces are given by $T(x, y, z) = k$ for constant k; that is, $x^2 + y^2 + z^2 = 100 - k$. If $100 - k > 0$, the graph of $x^2 + y^2 + z^2 = 100 - k$ is a sphere of radius $\sqrt{100 - k}$ and center $(0, 0, 0)$. When $k = 100$, the graph is a single point (the origin), and $T(0, 0, 0) = 100$°C. As the temperature drops, the constant k gets smaller and the radius $\sqrt{100 - k}$ of the sphere gets larger. Hence, the isothermal surfaces are spheres, and the larger the radius, the cooler the surface. This situation is illustrated in Figure 11.7.

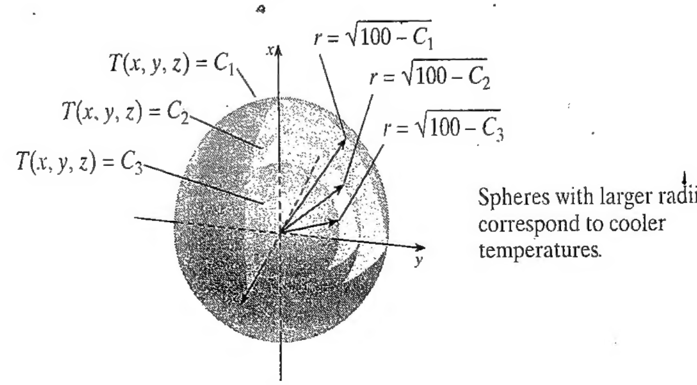


Figure 11.7 Isothermal surfaces for $T(x, y, z) = 100 - x^2 - y^2 - z^2$ ■

a circular
shown in

TECHNOLOGY NOTE

Computer software is now available for graphing functions of two variables. Software packages such as *Mathematica*, *Maple*, MATLAB, and *Derive* will sketch very sophisticated graphs in three dimensions. Software usually allows a given graph to be visualized from different viewpoints by rotating the surface until a desired viewpoint is found. A few examples of computer generated surfaces are shown in Figure 11.8.

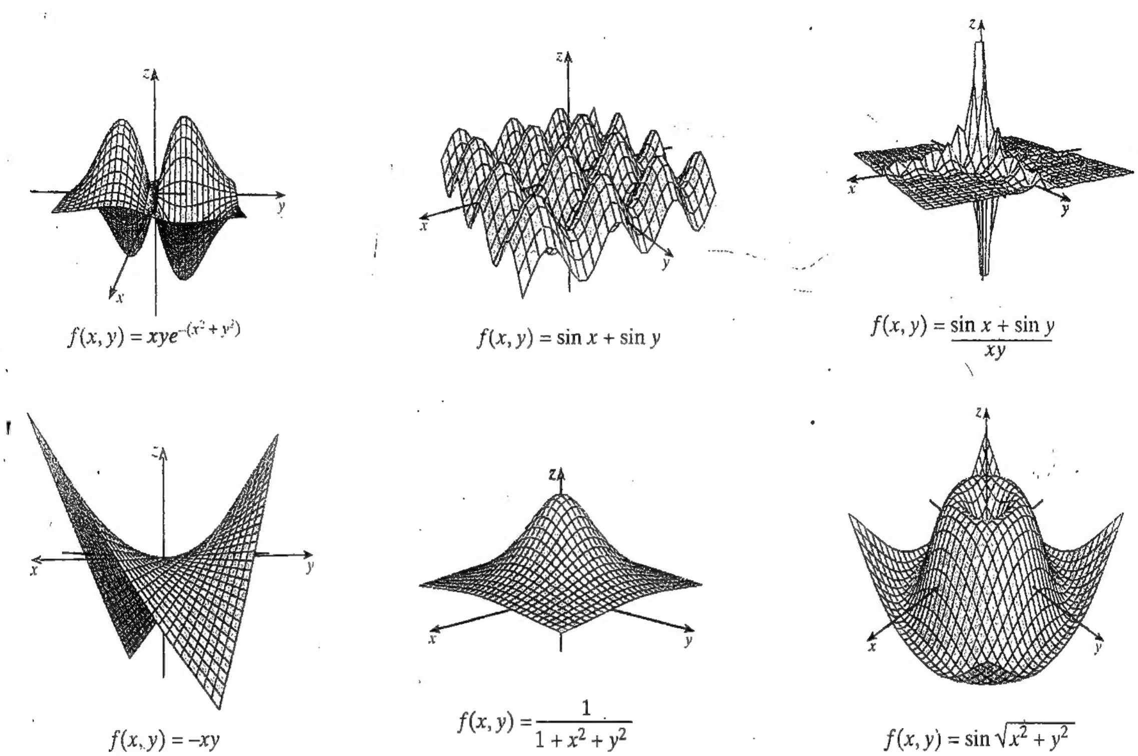


Figure 11.8 Several computer-generated surfaces

11.1 PROBLEM SET (HW copy pg1

Questions
§ 1–52

A 1. Let $f(x, y) = x^2y + xy^2$. If t is a real number, find:

a. $f(0, 0)$ b. $f(-1, 0)$ c. $f(0, -1)$
d. $f(1, 1)$ e. $f(2, 4)$ f. $f(t, t)$
g. $f(t, t^2)$ h. $(1 - t, t)$

2. Let $f(x, y, z) = x^2ye^{2x} + (x + y - z)^2$. Find

a. $f(0, 0, 0)$ b. $f(1, -1, 1)$ c. $f(-1, 1, -1)$
d. $\dfrac{d}{dx}f(x, x, x)$ e. $\dfrac{d}{dy}f(1, y, 1)$ f. $\dfrac{d}{dz}f(1, 1, z^2)$

3. **WHAT DOES THIS SAY?** Discuss what is meant by a function of two variables. Your discussion should include examples of a function of two variables not discussed in this section.

4. **WHAT DOES THIS SAY?** Describe how level curves can be used to sketch the graph of a function $f(x, y)$. Include some examples in your discussion (different from those in the text).

Find the domain and range for each function given in Problems 5–14.

5. $f(x, y) = \sqrt{x - y}$
6. $f(x, y) = \dfrac{1}{\sqrt{x - y}}$
7. $f(u, v) = \sqrt{uv}$
8. $f(x, y) = \sqrt{\dfrac{y}{x}}$
9. $f(x, y) = \ln(y - x)$
10. $f(u, v) = \sqrt{u \sin v}$
11. $f(x, y) = \sqrt{(x + 3)^2 + (y - 1)^2}$
12. $f(x, y) = e^{(x+1)/(y-2)}$
13. $f(x, y) = \dfrac{1}{\sqrt{x^2 - y^2}}$
14. $f(x, y) = \dfrac{1}{\sqrt{9 - x^2 - y^2}}$

Sketch some level curves $f(x, y) = C$ for $C \geq 0$ for each function given in Problems 15–20.

15. $f(x, y) = 2x - 3y$
16. $f(x, y) = x^2 - y^2$
17. $f(x, y) = x^3 - y$
18. $g(x, y) = x^2 - y$
19. $h(x, y) = x^2 + \dfrac{y^2}{4}$
20. $f(x, y) = \dfrac{x}{y}$

In Problems 21–26, sketch the level surface $f(x, y, z) = C$ for the given value of C.

21. $f(x, y, z) = y^2 + z^2$ for $C = 1$
22. $f(x, y, z) = x^2 + z^2$ for $C = 1$
23. $f(x, y, z) = x + y - z$ for $C = 1$

24. $f(x, y, z) = x + y - z$ for $C = 0$

25. $f(x, y, z) = (x + 1)^2 + (y - 2)^2 + (z - 3)^2$ for $C = 4$

26. $f(x, y, z) = 2x^2 + 2y^2 - z$ for $C = 1$

In Problems 27–34, describe the traces of the given quadric surface in each coordinate plane, then sketch the surface.

27. $9x^2 + 4y^2 + z^2 = 1$ 28. $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1$

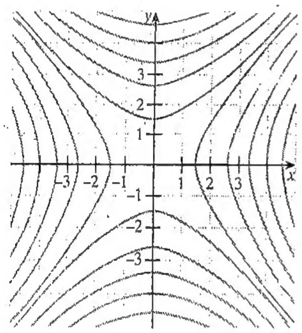
29. $\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$ 30. $\frac{x^2}{9} - y^2 - z^2 = 1$

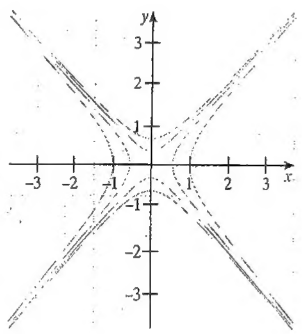
31. $z = x^2 + \frac{y^2}{4}$ 32. $z = \frac{x^2}{9} - \frac{y^2}{16}$

33. $x^2 + 2y^2 = 9z^2$ 34. $z^2 = 1 + \frac{x^2}{9} + \frac{y^2}{4}$

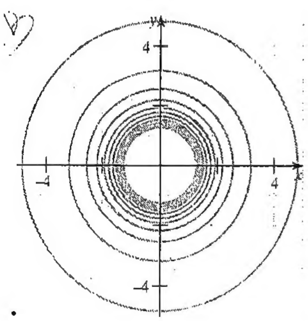
Match each family of level curves given in Problems 35–40 with one of the surfaces labeled A–F.

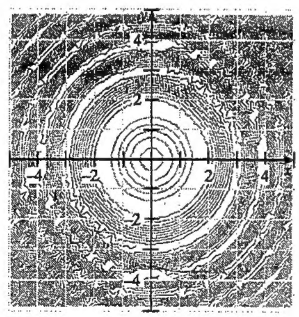
35. $f(x, y) = x^2 - y^2$ 36. $f(x, y) = e^{1-x^2+y^2}$



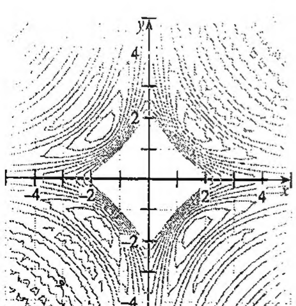


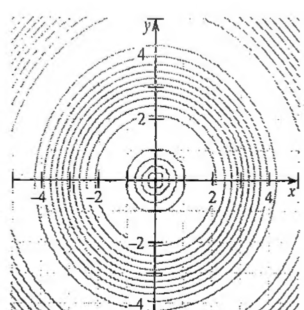
$f(x, y) = \frac{1}{x^2 + y^2}$ 38. $f(x, y) = \sin\left(\frac{x^2 + y^2}{2}\right)$

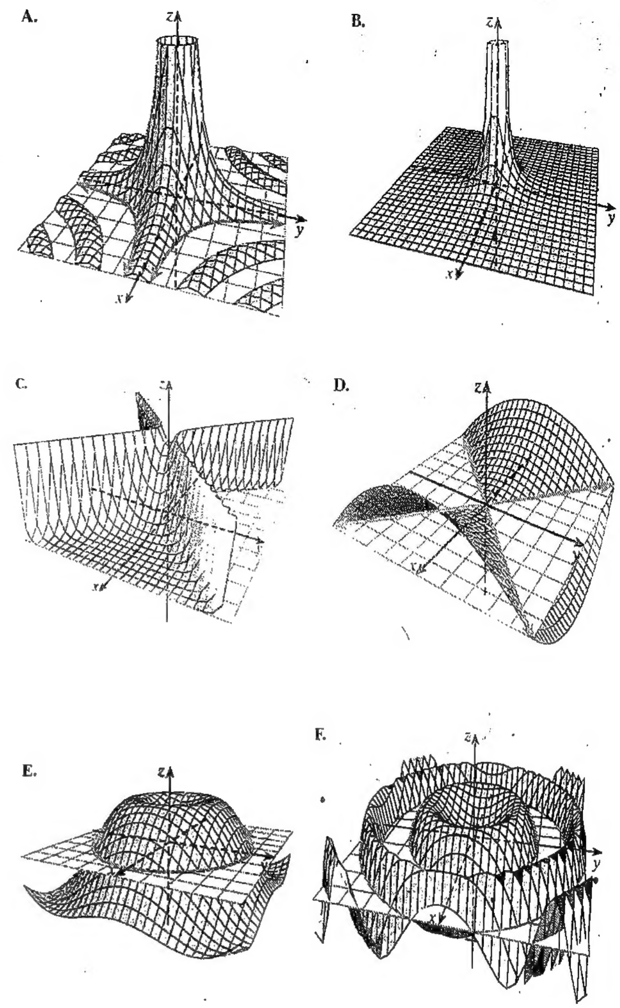




39. $f(x, y) = \frac{\cos xy}{x^2 + y^2}$ 40. $f(x, y) = \sin\sqrt{x^2 + y^2}$







B *Sketch the graph of each function given in Problems 41–52.*

41. $f(x, y) = -4$ 42. $f(x, y) = x$

43. $f(x, y) = y^2 + 1$ 44. $f(x, y) = x^3 - 1$

45. $f(x, y) = 2x - 3y$ 46. $f(x, y) = x^2 - y$

47. $f(x, y) = 2x^2 + y^2$ 48. $f(x, y) = x^2 - y^2$

49. $f(x, y) = \frac{x}{y}$ 50. $f(x, y) = \sqrt{x + y}$

51. $f(x, y) = x^2 + y^2 + 2$ 52. $f(x, y) = \sqrt{1 - x^2 - y^2}$

53. If $E(x, y)$ is the voltage (potential) at each point (x, y) in the plane, the level curves of E are called *equipotential curves.* Suppose

$$E(x, y) = \frac{7}{\sqrt{3 + x^2 + 2y^2}}$$

Sketch the equipotential curves that correspond to $E = 1$, $E = 2$, and $E = 3$.

54. According to the ideal gas law, $PV = kT$, where P is pressure, V is volume, T is temperature, and k is a constant. Suppose a tank contains 3,500 in.^{3} of a particular gas at a pressure of 24 lb/in.^{2} when the temperature is 270°K (degrees Kelvin).

a. Determine the constant of proportionality k for this gas.

b. Express the temperature T as a function of P and V using the constant found in part **a** and describe the isotherms for the temperature function.

55. The *lens*

where d lens, d_i i and L is as a fun function

56. The EZ worker- number A, a, a decides how th these c

a. a +

57. Suppos each d phones that re are fin

11.2

55. The *lens equation* in optics states that

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{L}$$

where d_o is the distance of an object from a thin, spherical lens, d_i is the distance of its image on the other side of the lens, and L is the *focal length* of the lens (see Figure 11.9). Write L as a function of d_o and d_i, and sketch some level curves of the function (these are curves of constant focal length).

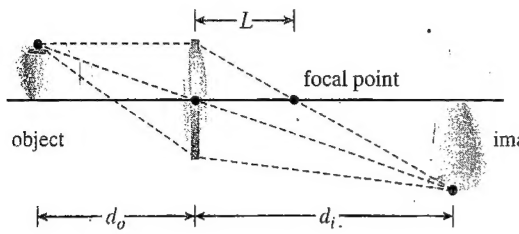


Figure 11.9 Image of an object through a lens

56. The EZGRO agricultural company estimates that when $100x$ worker-hours of labor are employed on y acres of land, the number of bushels of wheat produced is $f(x, y) = Ax^a y^b$, where A, a, and b are nonnegative constants. Suppose the company decides to double the production factors x and y. Determine how this decision affects the production of wheat in each of these cases:

a. $a + b > 1$ **b.** $a + b < 1$ **c.** $a + b = 1$

57. Suppose that when x machines and y worker-hours are used each day, a certain factory will produce $Q(x, y) = 10xy$ mobile phones. Describe the relationship between the "inputs" x and y that result in an "output" of 1,000 phones each day. *Note*: You are finding a level curve of the production function Q.

58. **Modeling Problem** At a certain factory, the daily output is modeled by $Q = CK^r L^{1-r}$ units, where K denotes capital investment, L is the size of the labor force, and C and r are constants, with $0 < r < 1$ (this is called a *Cobb-Douglas production function*). What happens to Q if K and L are both doubled? What if both are tripled?

59. Sketch the level curves for the graph of $f(x, y) = xy$ in the first quadrant.

60. A publishing house has found that in a certain city each of its salespeople will sell approximately

$$\frac{r^2}{2{,}000p} + \frac{s^2}{100} - s \quad \text{units}$$

per month, at a price of p dollars/unit, where s denotes the total number of salespeople employed and r is the amount of money spent each month on local advertising. Express the total revenue R as a function of p, r, and s.

61. **Modeling Problem** A manufacturer with exclusive rights to a sophisticated new industrial machine is planning to sell a limited number of the machines to both foreign and domestic firms. The price that the manufacturer can expect to receive for the machines will depend on the number of machines made available. It is estimated that if the manufacturer supplies x machines to the domestic market and y machines to the foreign market, the machines will sell for

$$60 - \frac{x}{5} + \frac{y}{20}$$

thousand dollars apiece at home and

$$50 - \frac{x}{10} + \frac{y}{20}$$

thousand dollars apiece abroad. Express the revenue R as a function of x and y.

11.2 Limits and Continuity

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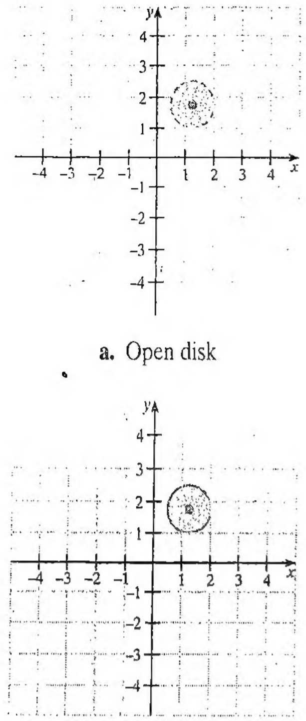
IN THIS SECTION open and closed sets in $\mathbb{R}^2$ and $\mathbb{R}^3$, limit of a function of two variables, continuity, limits and continuity for functions of three variables

Most commonly considered functions of a single variable have domains that can be described in terms of intervals. However, dealing with functions of two or more variables requires special terminology and notation that we will introduce in this section and then use to discuss limits and continuity for functions of two variables.

OPEN AND CLOSED SETS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

An **open disk** centered at the point $C(a, b)$ in $\mathbb{R}^2$ is the set of all points (x, y) such that

$$\boxed{\sqrt{(x-a)^2 + (y-b)^2} < r}$$



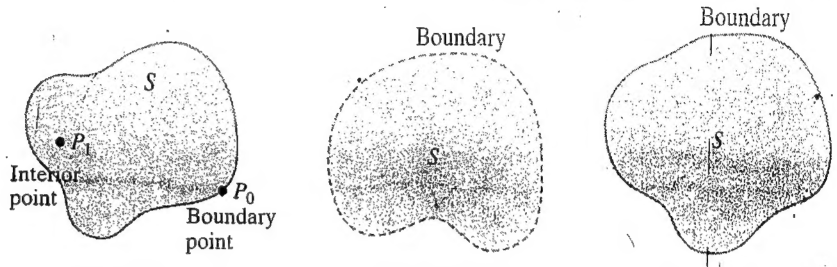
a. Open disk

b. Closed disk

Figure 11.10 Open and closed disks

for $r > 0$ (see Figure 11.10a). If the boundary of the disk is included (that is, if $\sqrt{(x-a)^2+(y-b)^2} \le r$), the disk is said to be **closed** (see Figure 11.10b). Open and closed disks are analogous to open and closed intervals on a coordinate line.

A point $P_0(x_0, y_0)$ is said to be an **interior point** of a set S in $\mathbb{R}^2$ if some open disk centered at P_0 is contained entirely within S, as shown in Figure 11.11a. If S is the empty set, or if every point of S is an interior point, then S is called an **open set** (Figure 11.11b). The point P_0 is called a **boundary point of** S if every open disk centered at P_0 contains both points that belong to S and points that do not. The collection of all boundary points of S is called the **boundary** of S, and S is said to be **closed** if it contains its boundary (Figure 11.11c). The empty set and $\mathbb{R}^2$ are both open and closed.



a. Interior points and boundary points of a set S.

b. The set S is open if all its points are interior points.

c. The set S is closed if it contains all its boundary points.

Figure 11.11 Open and closed sets in $\mathbb{R}^2$

Similarly, an **open ball** centered at $C(a, b, c)$ is the set of all points $P(x, y, z)$ such that

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < r$$

for $r > 0$. A point P_0 is an **interior point** of a set S in $\mathbb{R}^3$ if there exists an open ball centered at P_0 that is contained entirely within S, and the nonempty set S is an **open set** if all its points are interior points. The point P_0 is a **boundary point** of S if every open ball centered at P_0 contains both points that belong to S and points that do not, and S is **closed** if it contains all its boundary points. As with $\mathbb{R}^2$, the set $\mathbb{R}^3$ is both open and closed.

LIMIT OF A FUNCTION OF TWO VARIABLES

In Chapter 2, we informally defined the limit statement $\lim_{x\to c} f(x) = L$ to mean that $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close (but not equal) to c. For a function of two variables we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

to mean that the functional values $f(x, y)$ can be made arbitrarily close to the number L by choosing the point (x, y) sufficiently close to the point (x_0, y_0). The following is a more formal definition of this limiting behavior.

Limit of a Function of Two Variables

The limit statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

means that for each given number $\epsilon > 0$, there exists a number $\delta > 0$ so that whenever (x, y) is a point in the domain D of f such that

$$0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

then

$$|f(x,y) - L| < \epsilon$$

➡ **What This Says** If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L$, then given $\epsilon > 0$, the function value of $f(x, y)$ must lie in the interval $(L - \epsilon, L + \epsilon)$ whenever (x, y) is a point in the domain of f other than $P_0(x_0, y_0)$ that lies inside the disk of radius δ centered at P_0. This is illustrated in Figure 11.12.

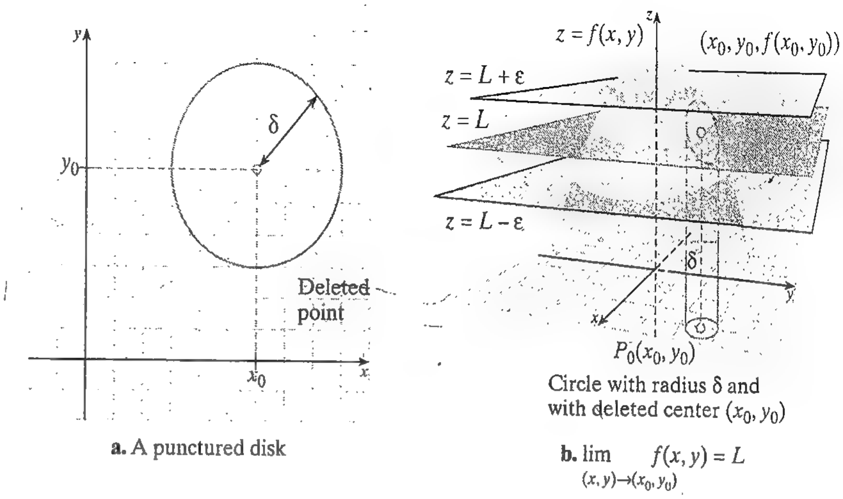


Figure 11.12 Limit of a function of two variables

When considering $\lim\limits_{x\to c} f(x)$ we need to examine the approach of x to c from two directions (the left- and right-hand limits). However, for a function of two variables, we write $(x, y) \to (x_0, y_0)$ to mean that the point (x, y) is allowed to approach (x_0, y_0) along any curve in the domain of f that passes through (x_0, y_0). For example, no matter how you get to the 15th step of Denver's state capitol, you will still be at 5,280 ft when you get there. If the

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

is not the same for all approaches, or **paths** within the domain of f, then *the limit does not exist.*

EXAMPLE 1 Evaluating a limit of a function of two variables

Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + x - xy - y}{x - y}$.

Solution

First, note that for $x \neq y$,

$$f(x, y) = \frac{x^2 + x - xy - y}{x - y} = \frac{(x + 1)(x - y)}{x - y} = x + 1$$

Therefore, since $f(x, y)$ is defined only for $x \neq y$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + x - xy - y}{x - y} = \lim_{(x,y)\to(0,0)} (x + 1) = 1$$ ■

EXAMPLE 2 Showing a limit does not exist

If $f(x, y) = \dfrac{2xy}{x^2 + y^2}$, show

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

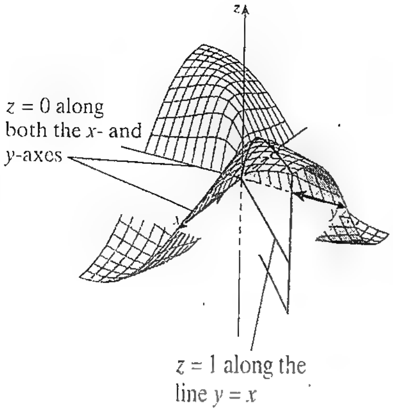


Figure 11.13 Graph of $z = \dfrac{2xy}{x^2+y^2}$ and limits as $(x, y) \to (0, 0)$

does not exist by evaluating this limit along the x-axis, the y-axis, and along the line $y = x$.

Solution

First note that the denominator is zero at (0, 0), so $f(0, 0)$ is not defined. If we approach the origin along the x-axis (where $y = 0$), we find that

$$f(x, 0) = \frac{2x(0)}{x^2+0} = 0$$

so $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along $y = 0$ (and $x \neq 0$). We find a similar result if we approach the origin along the y-axis (where $x = 0$); see Figure 11.13.

However, along the line $y = x$, the functional values are

$$f(x, y) = f(x, x) = \frac{2x^2}{x^2+x^2} = 1 \qquad \text{for} \quad x \neq 0$$

so that $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along $y = x$. Because $f(x, y)$ tends toward different numbers as $(x, y) \to (0, 0)$ along different curves, it follows that f has no limit at the origin. ■

EXAMPLE 3 Showing a limit does not exist

Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

Solution

If we approach this problem as we did the one in Example 2, we would take limits as $(x, y) \to (0, 0)$ along the x-axis, the y-axis, and the line $y = x$. All these approach lines have the general form $y = mx$. We find

along $y = mx$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2(mx)}{x^4+(mx)^2}$$
$$= \lim_{(x,y)\to(0,0)} \frac{mx}{x^2+m^2} = 0$$

However, if we approach (0, 0) along the parabola $y = x^2$, we find

along $y = x^2$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2(x^2)}{x^4+(x^2)^2}$$
$$= \lim_{(x,y)\to(0,0)} \frac{x^4}{2x^4} = \frac{1}{2}$$

Therefore, since approaching (0, 0) along the lines $y = mx$ gives a different limiting value from approaching along the parabola $y = x^2$, we conclude that the given limit does not exist. ■

WARNING It is often possible to show that a limit does not exist by the methods illustrated in Examples 2 and 3. However, it is impossible to try to prove that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exists by showing the limiting value of $f(x, y)$ is the same along every curve that passes through (x_0, y_0) because there are too many such curves.

We have just observed that it is difficult to show that a given limit exists. However, limits of functions of two variables that are known to exist can be manipulated in much the same way as limits involving functions of a single variable. Here is a list of the basic rules for manipulating such limits.

Basic Formulas and Rules for Limits of a Function of Two Variables

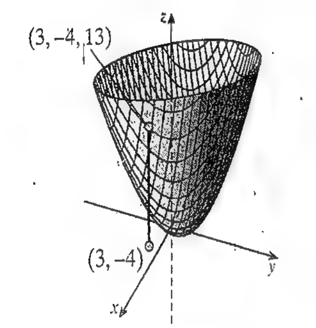
Suppose $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = M$. Then

Scalar multiple rule $\lim\limits_{(x,y)\to(x_0,y_0)} [af](x,y) = aL$ for constant a

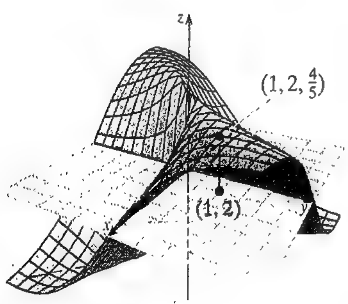
Sum rule $\lim\limits_{(x,y)\to(x_0,y_0)} [f+g](x,y) = L + M$

Product rule $\lim\limits_{(x,y)\to(x_0,y_0)} [fg](x,y) = LM$

Quotient rule $\lim\limits_{(x,y)\to(x_0,y_0)} \left[\dfrac{f}{g}\right](x,y) = \dfrac{L}{M}$ if $M \neq 0$



a. Graph of $z = x^2 + xy + y^2$



b. Graph of $f(x,y) = \dfrac{2xy}{x^2+y^2}$

Figure 11.14 Graphs showing limits

EXAMPLE 4 Manipulating limits of functions of two variables

Assuming each limit exists, evaluate:

a. $\lim\limits_{(x,y)\to(3,-4)} (x^2 + xy + y^2)$ **b.** $\lim\limits_{(x,y)\to(1,2)} \dfrac{2xy}{x^2+y^2}$

Solution

a. $\lim\limits_{(x,y)\to(3,-4)} (x^2 + xy + y^2) = (3)^2 + (3)(-4) + (-4)^2 = 13$. A graph is shown in Figure 11.14a.

b.
$$\lim_{(x,y)\to(1,2)} \frac{2xy}{x^2+y^2} = \frac{\lim\limits_{(x,y)\to(1,2)} 2xy}{\lim\limits_{(x,y)\to(1,2)} (x^2+y^2)}$$
$$= \frac{2(1)(2)}{1^2+2^2} = \frac{4}{5}$$

The graph is shown Figure 11.14b.

CONTINUITY

Recall that a function of a single variable x is continuous at $x = c$ if

1. $f(c)$ is defined; **2.** $\lim\limits_{x\to c} f(x)$ exists; **3.** $\lim\limits_{x\to c} f(x) = f(c)$.

Using the definition of the limit of a function of two variables, we can now define the continuity of a function of two variables analogously.

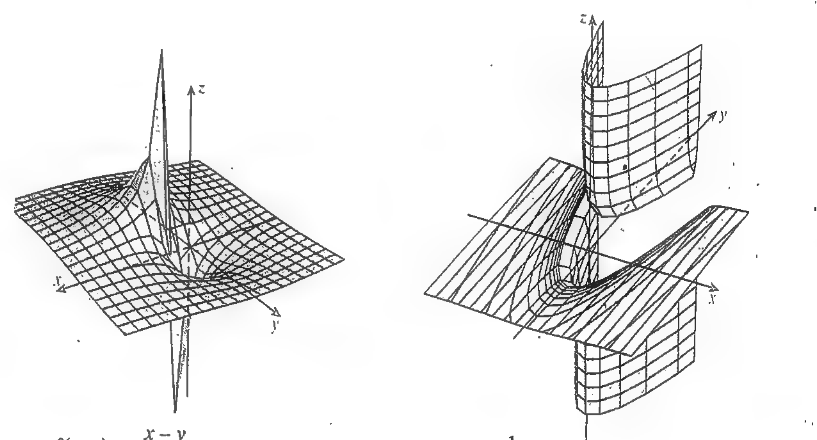
Continuity of a Function of Two Variables

The function $f(x, y)$ is **continuous** at the point (x_0, y_0) if

1. $f(x_0, y_0)$ is defined;
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists;
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

The function f is **continuous on a set** S if it is continuous at each point in S.

➡ **What This Says,** The function f is continuous at (x_0, y_0) if the functional value of $f(x, y)$ is close to $f(x_0, y_0)$ whenever (x, y) in the domain of f is sufficiently close to (x_0, y_0). Geometrically, this means that f is continuous if the surface $z = f(x, y)$ has no "gaps" or "holes" (see Figure 11.15).



a. $f(x, y) = \dfrac{x - y}{x^2 + y^2}$ is discontinuous at $(0, 0)$; a hole

b. $f(x, y) = \dfrac{1}{y - x^2}$ is discontinuous on the parabola $y = x^2$; a gap

Figure 11.15 Two types of discontinuity

The basic properties of limits can be used to show that if f and g are both continuous on the set S, then so are the sum $f + g$, the multiple af, the product fg, the quotient f/g (whenever $g \neq 0$), and the root $\sqrt[n]{f}$ wherever it is defined. Also, if F is a function of two variables that is continuous at (x_0, y_0) and G is a function of one variable that is continuous at $F(x_0, y_0)$, it can be shown that the composite function $G \circ F$ is continuous at (x_0, y_0).

Many common functions of two variables are continuous wherever they are defined. For instance, a polynomial in two variables, such as $x^3y^2 + 3xy^3 - 7x + 2$, is continuous throughout the plane, and a rational function of two variables is continuous wherever the denominator polynomial is not zero. Unless otherwise stated, the functions of two or more variables considered in this text will be continuous wherever they are defined.

EXAMPLE 5 Testing for continuity

Test the continuity of the functions:

a. $f(x, y) = \dfrac{x - y}{x^2 + y^2}$ **b.** $f(x, y) = \dfrac{1}{y - x^2}$

These are the functions whose graphs are shown in Figure 11.15.

Solution

a. The function f is a rational function of x and y (since $x - y$ and $x^2 + y^2$ are both polynomials), so it is discontinuous only where it is undefined; namely at (0, 0).
b. Again, this is a rational function and is discontinuous only where it is undefined; that is, where

$$y - x^2 = 0$$

Thus, the function is continuous at all points except those lying on the parabola $y = x^2$. ■

EXAMPLE 6 Continuity using the limit definition

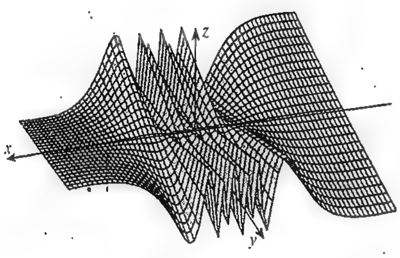
Show that f is continuous at (0, 0) where

$$f(x, y) = \begin{cases} y \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

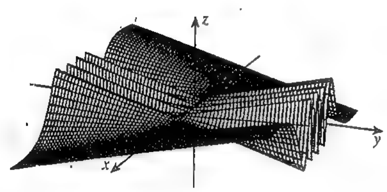
Solution

The graph is shown in Figure 11.16. To prove continuity at (0, 0), we must show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - f(0, 0)| = \left| y \sin \frac{1}{x} \right| < \epsilon \quad \text{whenever} \quad 0 < x^2 + y^2 < \delta^2$$



Front view



Side view

Figure 11.16 Graph of f in Example 6

(We use $x^2 + y^2$ and δ^2 here instead of $\sqrt{x^2 + y^2}$ and δ for convenience.) Note that $\left| y \sin \frac{1}{x} \right| \leq |y|$ for all $x \neq 0$, because $\left| \sin \frac{1}{x} \right| \leq 1$ for $x \neq 0$. If (x, y) lies in the disk $x^2 + y^2 < \delta^2$, then the points $(0, y)$ that satisfy $y^2 < \delta^2$ lie in the same disk (let $x = 0$ in $x^2 + y^2 < \delta^2$). In other words, points satisfying $|y| < \delta$ lie in the disk, and if we let $\delta = \epsilon$ it follows that

$$|f(x, y) - f(0, 0)| \leq |y| < \delta = \epsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad ■$$

LIMITS AND CONTINUITY FOR FUNCTIONS OF THREE VARIABLES

The concepts we have just introduced for functions of two variables in $\mathbb{R}^2$ extend naturally to functions of three variables in $\mathbb{R}^3$. In particular, the limit statement

$$\lim_{(x,y,z) \to (x_0, y_0, z_0)} f(x, y, z) = L$$

means that for each number $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x, y, z) - L| < \epsilon$$

whenever (x, y, z) is a point in the domain of f such that

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta$$

The function $f(x, y, z)$ is **continuous** at the point $P_0(x_0, y_0, z_0)$ if

1. $f(x_0, y_0, z_0)$ is defined;
2. $\displaystyle\lim_{(x,y,z) \to (x_0, y_0, z_0)} f(x, y, z)$ exists;
3. $\displaystyle\lim_{(x,y,z) \to (x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0)$.

Most commonly considered functions of three variables are continuous wherever they are defined. We close with an example that illustrates how to determine the set of discontinuities of a function of three variables.

Note

EXAMPLE 7 Determining where a function of three variables is continuous

For what points (x, y, z) is the following function continuous?

$$f(x, y, z) = \frac{3}{\sqrt{x^2 + y^2 - 2z}}$$

Solution

The function $f(x, y, z)$ is continuous except where it is not defined; that is, for $x^2 + y^2 - 2z \leq 0$. Thus, $f(x, y, z)$ is continuous at any point not inside or on the paraboloid $z = \frac{1}{2}(x^2 + y^2)$.

11.2 PROBLEM SET (pg 32-51) Q 1-50 Guidelines (except Q 41)

A

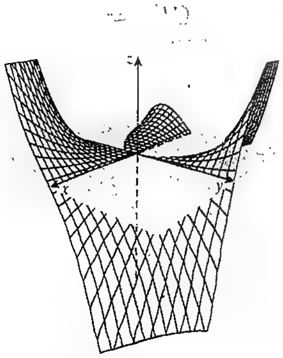
1. **WHAT DOES THIS SAY?** Describe the notion of a limit of a function of two variables.
2. **WHAT DOES THIS SAY?** Describe the basic formulas and rules for limits of a function of two variables.

In Problems 3–20, find the given limit, assuming it exists.

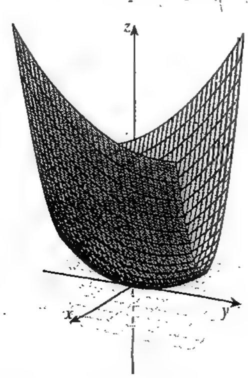
3. $\displaystyle\lim_{(x,y)\to(-1,0)} (xy^2 + x^3y + 5)$
4. $\displaystyle\lim_{(x,y)\to(0,0)} (5x^2 - 2xy + y^2 + 3)$
5. $\displaystyle\lim_{(x,y)\to(1,3)} \frac{x+y}{x-y}$
6. $\displaystyle\lim_{(x,y)\to(3,4)} \frac{x-y}{\sqrt{x^2+y^2}}$
7. $\displaystyle\lim_{(x,y)\to(1,0)} e^{xy}$
8. $\displaystyle\lim_{(x,y)\to(1,0)} (x+y)e^{xy}$
9. $\displaystyle\lim_{(x,y)\to(0,1)} [e^{x^2+y}\ln(ey^2)]$
10. $\displaystyle\lim_{(x,y)\to(e,0)} \ln(x^2+y^2)$
11. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - 2xy + y^2}{x - y}$
12. $\displaystyle\lim_{(x,y)\to(1,2)} \frac{(x^2-1)(y^2-4)}{(x-1)(y-2)}$
13. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^x \tan^{-1} y}{y}$
14. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$
15. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y}$
16. $\displaystyle\lim_{(x,y)\to(0,0)} (\sin x + \cos y)$
17. $\displaystyle\lim_{(x,y)\to(5,5)} \frac{x^4 - y^4}{x^2 - y^2}$
18. $\displaystyle\lim_{(x,y)\to(a,a)} \frac{x^4 - y^4}{x^2 - y^2}$; a is a constant.
19. $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x^2 - 4y^2}{x - 2y}$
20. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+4}-2}$

In Problems 21–29, evaluate the indicated limit, if it exists.

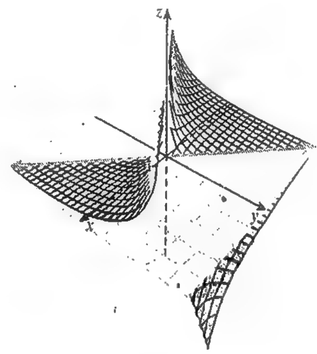
21. $\displaystyle\lim_{(x,y)\to(2,1)} (xy^2 + x^3y)$



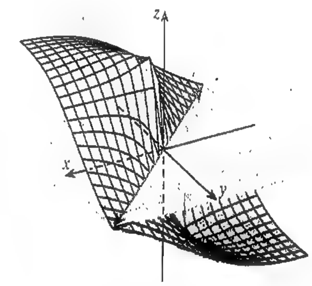
22. $\displaystyle\lim_{(x,y)\to(1,2)} (5x^2 - 2xy + y^2)$



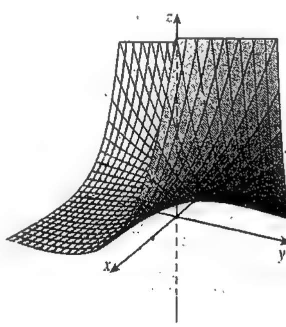
23. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$



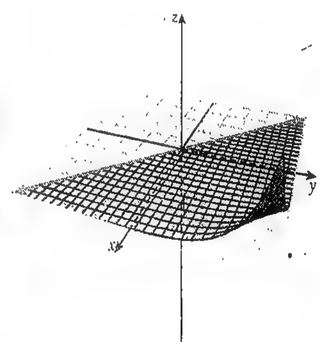
24. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x^2+y^2}}$



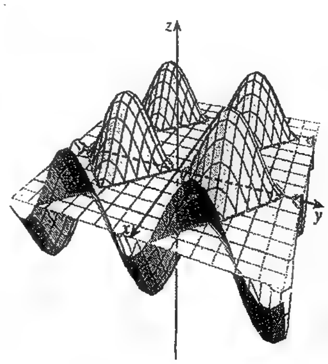
25. $\displaystyle\lim_{(x,y)\to(0,0)} e^{xy}$



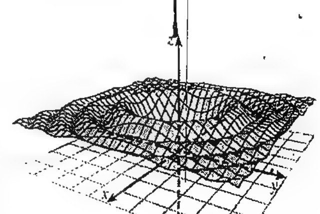
26. $\displaystyle\lim_{(x,y)\to(1,1)} (x+y)e^{xy}$



27. $\displaystyle\lim_{(x,y)\to(0,0)} (\sin x - \cos y)$



28. $\displaystyle\lim_{(x,y)\to(0,0)} \left[1 - \frac{\sin(x^2+y^2)}{x^2+y^2}\right]$



29. $\displaystyle\lim_{(x,y)\to(0,0)}$

In Problems

30. $f(x, y)$

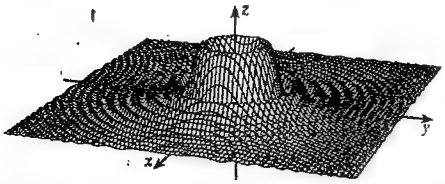
31. $f(x, y$

32. $f(x,$

33. $f(x,$

ntinuous

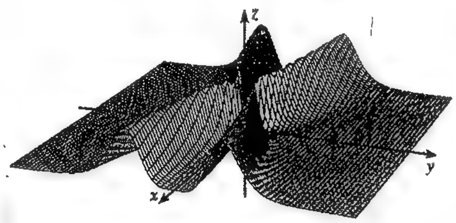
29. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$



x^2+y^2-
paraboloid

B *In Problems 30–33, show that* $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ *does not exist.*

30. $f(x,y)=\dfrac{x^4y^4}{(x^2+y^4)^3}$



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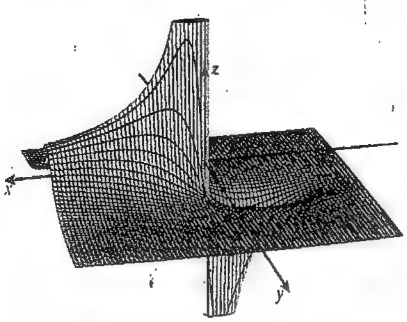
31. $f(x,y)=\dfrac{x-y^2}{x^2+y^2}$



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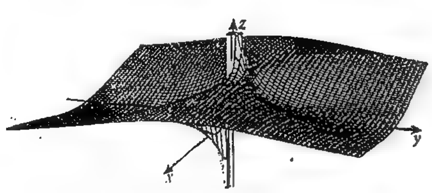
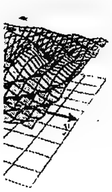
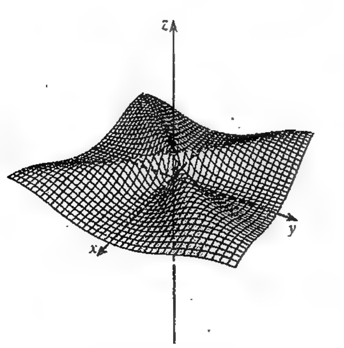
32. $f(x,y)=\dfrac{x^2+y}{x^2+y^2}$



$(x^2+y^2)]$
$^2+y^2]$

33. $f(x,y)=\dfrac{x^2y^2}{x^4+y^4}$





34. Let f be the function defined by

$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4} & \text{for } (x,y)\neq(0,0)\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$$

Is f continuous at (0, 0)? Explain.

HW pg 4 Ans obtained correct (at back wrong)

35. Let f be the function defined by

$$f(x,y)=\begin{cases}\dfrac{xy^3}{x^2+y^6} & \text{for } (x,y)\neq(0,0)\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$$

Is f continuous at (0, 0)? Explain.

36. Let f be the function defined by $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ for $(x,y)\neq(0,0)$.

a. Find $\displaystyle\lim_{(x,y)\to(2,1)} f(x,y)$.

b. Prove that f has no limit at (0, 0).

37. Let f be the function defined by $f(x,y)=\dfrac{x^2+2y^2}{x^2+y^2}$ for $(x,y)\neq(0,0)$.

a. Find $\displaystyle\lim_{(x,y)\to(3,1)} f(x,y)$.

b. Prove that f has no limit at (0, 0).

38. Given that the function

$$f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\\ A & \text{for } (x,y)=(0,0)\end{cases}$$

is continuous at the origin, what is A?

39. Given that the function

$$f(x,y)=\begin{cases}\dfrac{3x^3-3y^3}{x^2-y^2} & \text{for } x^2\neq y^2\\ B & \text{otherwise}\end{cases}$$

is continuous at the origin, what is B?

40. Let

$$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$$

HW See pg 76

Given that $f(x,y)$ has a limit at (0, 0), is f continuous there?

41. Assuming that the limit exists, show that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}=0$$

do by spherical coord. approach

42. Let $f(x,y)=\dfrac{\sin(x^2+y^2)}{x^2+y^2}$ for $(x,y)\neq(0,0)$. For what value of $f(0,0)$ is $f(x,y)$ continuous at (0, 0)? *Hint*: Use polar coordinates.

43. Consider the function

$$f(x,y)=\begin{cases}\dfrac{2x^2-x^2y^2+2y^2}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\\ A & \text{for } (x,y)=(0,0)\end{cases}$$

Find the value of A which will make f continuous at the origin. *Hint*: Use polar coordinates.

Use polar coordinates to find the limit given in Problems 44–47.

44. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2}$

45. $\displaystyle\lim_{(x,y)\to(0,0)} (1+x^2+y^2)^{1/(x^2+y^2)}$

46. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$

47. $\displaystyle\lim_{(x,y)\to(0,0)} x\ln\sqrt{x^2+y^2}$

Counterexample Problems *In Problems 48–50, either show that the given statement is true or find a counterexample.*

48. If $\lim_{y\to 0} f(0,y)=0$, then $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)=0$.

Not true. Let $f(x,y)=\frac{xy}{x^2+y^2}$. Then $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist even though $\lim_{y\to 0} f(0,y)=0$

49. If $f(x,y)$ is continuous for all $x\neq 0$ and $y\neq 0$, and $f(0,0)=0$, then $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)=0$.

If doesn't exist e.g. $f(x,y)=\begin{cases}\frac{x-y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0; & \text{elsewhere}\end{cases}$

50. If $f(x)$ and $g(y)$ are continuous functions of x and y, respectively, then

$$h(x,y)=f(x)+g(y)$$

is a continuous function of x and y.

True. $\displaystyle\lim_{(x,y)\to(a,b)} h(x,y) = \lim_{(x,y)\to(a,b)} (f(x)+g(y)) = f(a)+g(b) = h(a,b)$

C *Use the ϵ-δ definition of limit to verify the limit statements given in Problems 51–54.*

HW Pg 76

51. $\displaystyle\lim_{(x,y)\to(0,0)} (2x^2+3y^2)=0$

52. $\displaystyle\lim_{(x,y)\to(0,0)} (x+y^2)=0$

Let $\varepsilon>0$ be given
Consider
$|f(x,y)-0| = |x+y^2-0| = |x+y^2| \le |x|+|y^2| \to (1)$
choose $\delta < \min\{1, \frac{\varepsilon}{2}\}$.
Now for $(x,y)\in D$ where $D=\{(x,y)\in\mathbb{R}^2 \mid \sqrt{x^2+y^2}<\delta\}$
$|x|<\delta$ & $|y|<\delta$
using in (1) : $|f(x,y)-0| < 2\delta < \varepsilon$ whenever $0<\sqrt{x^2+y^2}<\delta$
∴ proved.

$y<\delta<1$ so $y^2<y$

53. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}=0$

54. $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^2-y^2}{x+y}=2$

55. Prove that if f is continuous and $f(a,b)>0$, then there exists a δ-neighborhood about (a,b) such that $f(x,y)>0$ for every point (x,y) in the neighborhood.

56. Prove the scalar multiple rule:

$$\lim_{(x,y)\to(x_0,y_0)} [af](x,y) = a \lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

57. Prove the sum rule:

$$\lim_{(x,y)\to(x_0,y_0)} [f+g](x,y) = L+M$$

where $L=\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ and $M=\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x,y)$.

58. A function of two variables $f(x,y)$ may be continuous in each separate variable at $x=x_0$ and $y=y_0$ without being itself continuous at (x_0,y_0). Let $f(x,y)$ be defined by

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\\[2mm] 0 & \text{for } (x,y)=(0,0)\end{cases}$$

Let $g(x)=f(x,0)$ and $h(y)=f(0,y)$. Show that both $g(x)$ and $h(y)$ are continuous at 0 but that $f(x,y)$ is not continuous at $(0,0)$.

11.3 Partial Derivatives

no need to see CN copy

IN THIS SECTION partial differentiation, partial derivative as a slope, partial derivative as a rate, higher-order partial derivatives

PARTIAL DIFFERENTIATION

It is often important to know how a function of two variables changes with respect to one of the variables. For example, according to the ideal gas law, the pressure of a gas is related to its temperature and volume by the formula $P=\dfrac{kT}{V}$, where k is a constant. If the temperature is kept constant while the volume is allowed to vary, we might want to know the effect on the rate of change of pressure. Similarly, if the volume is kept constant while the temperature is allowed to vary, we might want to know the effect on the rate of change of pressure.

The process of differentiating a function of several variables with respect to one of its variables while keeping the other variable(s) fixed is called **partial differentiation**, and the resulting derivative is a **partial derivative** of the function.

Recall that the derivative of a function of a single variable f is defined to be the limit of a difference quotient, namely,

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Partial derivatives with respect to x or y are defined similarly.

Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the partial derivatives of f with respect to x and y are the functions f_x and f_y, respectively, defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist.

⇛ **What This Says** For the partial differentiation of a function of two variables, $z = f(x, y)$, we find the partial derivative with respect to x by regarding y as constant while differentiating the function with respect to x. Similarly, the partial derivative with respect to y is found by regarding x as constant while differentiating with respect to y.

EXAMPLE 1 Partial derivatives

If $f(x, y) = x^3y + x^2y^2$, find: **a.** f_x **b.** f_y

Solution

a. For f_x, hold y constant and find the derivative with respect to x:

$$f_x(x, y) = 3x^2y + 2xy^2$$

b. For f_y, hold x constant and find the derivative with respect to y:

$$f_y(x, y) = x^3 + 2x^2y$$ ■

Several different symbols are used to denote partial derivatives, as indicated in the following box.

Alternative Notation for Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = z_x = D_x(f)$$

and

$$f_y(x, y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = z_y = D_y(f)$$

The values of the partial derivatives of $f(x, y)$ at the point (a, b) are denoted by

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a, b)$$

EXAMPLE 2 Finding and evaluating a partial derivative

Let $z = x^2 \sin(3x + y^3)$.

a. Evaluate $\left.\dfrac{\partial z}{\partial x}\right|_{(\pi/3,0)}$. **b.** Evaluate z_y at $(1, 1)$.

Solution

a. $$\frac{\partial z}{\partial x} = 2x\sin(3x+y^3) + x^2\cos(3x+y^3)(3)$$
$$= 2x\sin(3x+y^3) + 3x^2\cos(3x+y^3)$$

Thus,

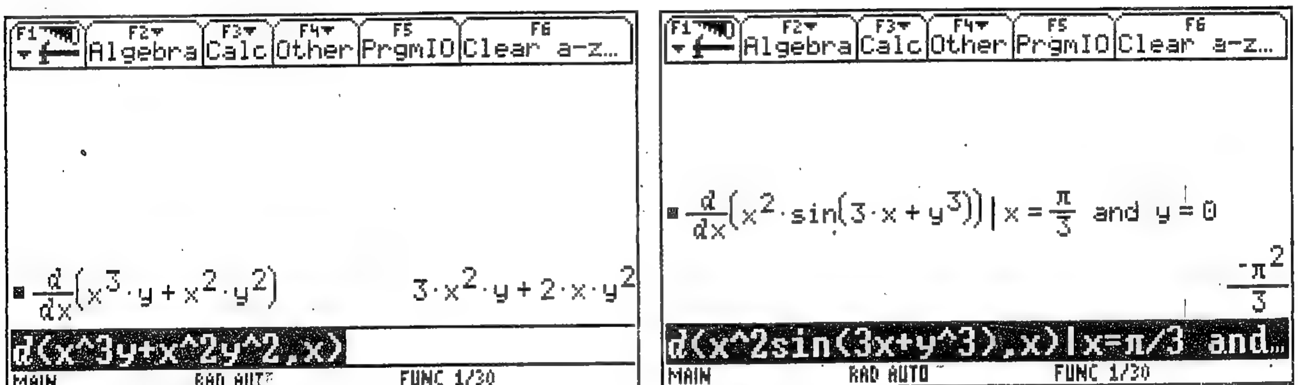
$$\left.\frac{\partial z}{\partial x}\right|_{(\pi/3,0)} = 2\left(\frac{\pi}{3}\right)\sin\pi + 3\left(\frac{\pi}{3}\right)^2\cos\pi = \frac{2\pi}{3}(0) + \frac{\pi^2}{3}(-1) = -\frac{\pi^2}{3}$$

b. $z_y = x^2\cos(3x+y^3)(3y^2) = 3x^2y^2\cos(3x+y^3)$ so that

$$z_y(1,1) = 3(1)^2(1)^2\cos(3+1) = 3\cos 4$$

TECHNOLOGY NOTE

Finding partial derivatives using technology is a natural extension of the way you have been finding other derivatives. The general format for most calculators and computer programs is the same: *derivative operator, function, variable of differentiation.* Evaluating the partial derivative is then accomplished by using the evaluate feature. For example, Figure 11.17a displays the computation of the partial derivative of $f(x, y) = x^3y + x^2y^2$ with respect to x from Example 1, while Figure 11.17b displays the evaluation of the partial derivative of $z = x^2\sin(3x + y^3)$ with respect to x at the point $(\frac{\pi}{3}, 0)$ found in Example 2.



a. The partial derivative $\frac{\partial}{\partial x}(x^3y + x^2y^2)$ b. The partial derivative $\frac{\partial}{\partial x}(x^2\sin(3x + y^3))$ evaluated at $(\frac{\pi}{3}, 0)$

Figure 11.17 Computing partial derivatives with technology

EXAMPLE 3 Partial derivative of a function of three variables

Let $f(x, y, z) = x^2 + 2xy^2 + yz^3$; determine: **a.** f_x **b.** f_y **c.** f_z

Solution

a. For f_x, think of f as a function of x alone with y and z treated as constants:

$$f_x(x, y, z) = 2x + 2y^2$$

b. $f_y(x, y, z) = 4xy + z^3$
c. $f_z(x, y, z) = 3yz^2$

WARNING In $f(x, y, z)$, z is an independent variable.

EXAMPLE 4 Partial derivative of an implicitly defined function

Let z be defined implicitly as a function of x and y by the equation

$$x^2z + yz^3 = x$$

Determine $\partial z/\partial x$ and $\partial z/\partial y$.

Solution

Differentiate implicitly with respect to x, treating y as a constant:

$$2xz + x^2\frac{\partial z}{\partial x} + 3yz^2\frac{\partial z}{\partial x} = 1$$

Then solve this equation for $\dfrac{\partial z}{\partial x}$:

$$\frac{\partial z}{\partial x} = \frac{1 - 2xz}{x^2 + 3yz^2}$$

Similarly, holding x constant and differentiating implicitly with respect to y, we find

$$x^2\frac{\partial z}{\partial y} + z^3 + 3yz^2\frac{\partial z}{\partial y} = 0$$

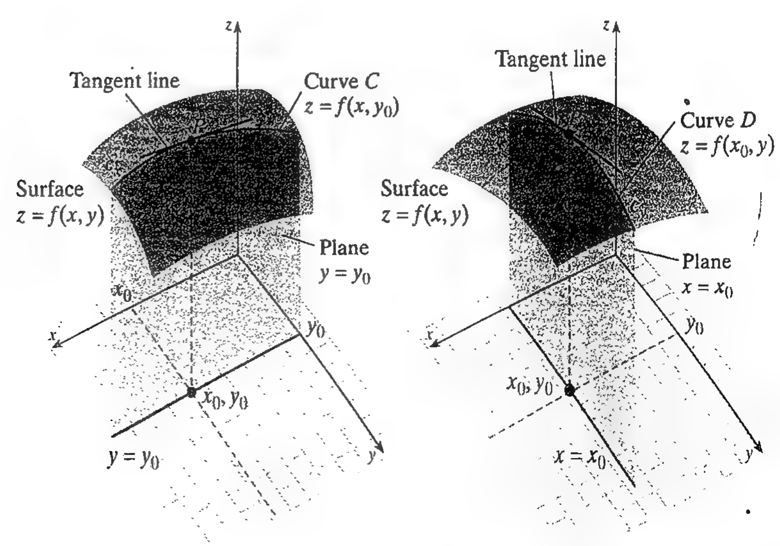
so that

$$\frac{\partial z}{\partial y} = \frac{-z^3}{x^2 + 3yz^2}$$

■

PARTIAL DERIVATIVE AS A SLOPE

A useful geometric interpretation of partial derivatives is indicated in Figure 11.18. In Figure 11.18a, the plane $y = y_0$ intersects the surface $z = f(x, y)$ in a curve C parallel to the xz-plane. That is, C is the trace of the surface in the plane $y = y_0$. An equation for this curve is $z = f(x, y_0)$, and because y_0 is fixed, the function depends only on x. Thus, we can compute the slope of the tangent line to C at the point $P(x_0, y_0, z_0)$ in the plane $y = y_0$ by differentiating $f(x, y_0)$ with respect to x and evaluating the derivative at $x = x_0$. That is, the slope is $f_x(x_0, y_0)$, the value of the partial derivative f_x at (x_0, y_0). The analogous interpretation for $f_y(x_0, y_0)$ is shown in Figure 11.18b.



a. The tangent line in the plane $y = y_0$ to the curve C at the point P has slope $f_x(x_0, y_0)$.

b. The tangent line in the plane $x = x_0$ to the curve D at the point P has slope $f_y(x_0, y_0)$.

Figure 11.18 Slope interpretation of the partial derivative

Partial Derivative as the Slope of a Tangent Line

The line parallel to the xz-plane and tangent to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0)$ has slope $f_x(x_0, y_0)$. Likewise, the tangent line to the surface at P_0 that is parallel to the yz-plane has slope $f_y(x_0, y_0)$.

EXAMPLE 5 Slope of a line parallel to the xz-plane

Find the slope of the line that is parallel to the xz-plane and tangent to the surface $z = x\sqrt{x+y}$ at the point $P(1, 3, 2)$.

Solution

If $f(x, y) = x\sqrt{x+y} = x(x+y)^{1/2}$, then the required slope is $f_x(1, 3)$.

$$f_x(x, y) = x\left(\frac{1}{2}\right)(x+y)^{-1/2}(1+0) + (1)(x+y)^{1/2} = \frac{x}{2\sqrt{x+y}} + \sqrt{x+y}$$

Thus, $f_x(1, 3) = \dfrac{1}{2\sqrt{1+3}} + \sqrt{1+3} = \dfrac{9}{4}$. ■

Higher-Order Derivatives

PARTIAL DERIVATIVE AS A RATE

The derivative of a function of one variable can be interpreted as a rate of change, and the analogous interpretation of partial derivative may be described as follows.

Partial Derivatives as Rates of Change

As the point (x, y) moves from the fixed point $P_0(x_0, y_0)$, the function $f(x, y)$ changes at a rate given by $f_x(x_0, y_0)$ in the direction of the positive x-axis and by $f_y(x_0, y_0)$ in the direction of the positive y-axis.

EXAMPLE 6 Partial derivatives as rates of change

In an electrical circuit with electromotive force (EMF) of E volts and resistance R ohms, the current is $I = E/R$ amperes. Find the partial derivatives $\partial I/\partial E$ and $\partial I/\partial R$ at the instant when $E = 120$ and $R = 15$ and interpret these derivatives as rates.

Solution

Since $I = ER^{-1}$, we have

$$\frac{\partial I}{\partial E} = R^{-1} \quad \text{and} \quad \frac{\partial I}{\partial R} = -ER^{-2}$$

and thus, when $E = 120$ and $R = 15$, we find that

$$\frac{\partial I}{\partial E} = 15^{-1} \approx 0.0667 \quad \text{and} \quad \frac{\partial I}{\partial R} = -(120)(15)^{-2} \approx -0.5333$$

This means that if the resistance is fixed at 15 ohms, the current is increasing (because the derivative is positive) with respect to voltage at the rate of 0.0667 ampere per volt when the EMF is 120 volts. Likewise, with the same fixed EMF, the current is decreasing (because the derivative is negative) with respect to resistance at the rate of 0.5333 ampere per ohm when the resistance is 15 ohms. ■

HIGHER-ORDER PARTIAL DERIVATIVES

The partial derivative of a function is a function; so it is possible to take the partial derivative of a partial derivative. This is very much like taking the second derivative of a function of one variable if we take two consecutive partial derivatives with respect to the same variable, and the resulting derivative is called the **second-order partial derivative** with respect to that variable. However, we can also take the partial derivative with respect to one variable and then take a second partial derivative with respect to a different variable, producing what is called a **mixed second-order partial derivative**. The higher-order partial derivatives for a function of two variables $f(x, y)$ are denoted as indicated in the following box:

Higher-Order Partial Derivatives

Given $z = f(x, y)$.

Second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy}$$

Mixed second-order partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$$

WARNING The notation f_{xy} means that we differentiate first with respect to x and then with respect to y, while $\dfrac{\partial^2 f}{\partial x \partial y}$ means just the opposite (differentiate with respect to y first and then with respect to x).

EXAMPLE 7 Higher-order partial derivatives of a function of two variables

For $z = f(x, y) = 5x^2 - 2xy + 3y^3$, determine these higher-order partial derivatives.

a. $\dfrac{\partial^2 z}{\partial x \partial y}$ **b.** $\dfrac{\partial^2 f}{\partial y \partial x}$ **c.** $\dfrac{\partial^2 z}{\partial x^2}$ **d.** $f_{xy}(3, 2)$

Solution

a. First differentiate with respect to y; then differentiate with respect to x.

$$\frac{\partial z}{\partial y} = -2x + 9y^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-2x + 9y^2) = -2$$

b. Differentiate first with respect to x and then with respect to y:

$$\frac{\partial f}{\partial x} = 10x - 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(10x - 2y) = -2$$

c. Differentiate with respect to x twice:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(10x - 2y) = 10$$

d. Evaluate the mixed partial found in part **b** at the point (3, 2):

$$f_{xy}(3, 2) = -2$$

■

Notice from parts **a** and **b** of Example 7 that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$. This equality of mixed partials does not hold for all functions, but for most functions we will encounter, it will be true. The following theorem provides sufficient conditions for this equality to occur.

THEOREM 11.1 Equality of mixed partials

If the function $f(x, y)$ has mixed second-order partial derivatives f_{xy} and f_{yx} that are continuous in an open disk containing (x_0, y_0), then

$$f_{yx}(x_0, y_0) = f_{xy}(x_0, y_0)$$

Proof This proof is omitted. □

EXAMPLE 8 Partial derivatives of functions of two variables

Determine f_{xy}, f_{yx}, f_{xx}, and f_{xxy}, where $f(x, y) = x^2 y e^y$.

Solution

We have the partial derivatives

$$f_x = 2xye^y \qquad f_y = x^2e^y + x^2ye^y$$

The mixed partial derivatives (which must be the same by the previous theorem) are

$$f_{xy} = (f_x)_y = 2xe^y + 2xye^y \qquad f_{yx} = (f_y)_x = 2xe^y + 2xye^y$$

Finally, we compute the second- and higher-order partial derivatives:

$$f_{xx} = (f_x)_x = 2ye^y \quad \text{and} \quad f_{xxy} = (f_{xx})_y = 2e^y + 2ye^y$$

■

An equation involving partial derivatives is called a **partial differential equation**. An important partial differential equation is the *diffusion* or *heat equation*

$$\frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial x^2}$$

where $T(x, t)$ is the temperature in a thin rod at position x and time t. The constant c is called the *diffusivity* of the material in the rod. In the following example, we verify that a certain function satisfies this heat equation.

EXAMPLE 9 Verifying that a function satisfies the heat equation

Verify that $T(x, t) = e^{-t} \cos \dfrac{x}{c}$ satisfies the heat equation, $\dfrac{\partial T}{\partial t} = c^2 \dfrac{\partial^2 T}{\partial x^2}$.

Solution

$$\frac{\partial T}{\partial t} = -e^{-t} \cos \frac{x}{c}$$

11.3

A 1. WH

2. Expl of t

Determ

3. f(x,

4. f(x,

5. f(x

and

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{1}{c}e^{-t}\sin\frac{x}{c}\right)$$
$$= -\frac{1}{c^2}e^{-t}\cos\frac{x}{c}$$

Thus, T satisfies the heat equation $\dfrac{\partial T}{\partial t} = c^2\dfrac{\partial^2 T}{\partial x^2}$. ■

Analogous definitions can be made for functions of more than two variables. For example,

$$f_{zzz} = \frac{\partial^3 f}{\partial z^3} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)\right] \quad \text{or} \quad f_{xyz} = \frac{\partial^3 f}{\partial z \partial y \partial x} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]$$

EXAMPLE 10 Higher-order partial derivatives of a function of several variables

By direct calculation, show that $f_{xyz} = f_{yzx} = f_{zyx}$ for the function $f(x, y, z) = xyz + x^2y^3z^4$.

Solution

First, compute the partials:

$$f_x(x, y, z) = yz + 2xy^3z^4$$
$$f_y(x, y, z) = xz + 3x^2y^2z^4$$
$$f_z(x, y, z) = xy + 4x^2y^3z^3$$

Next, determine the mixed partials:

$$f_{xy}(x, y, z) = (yz + 2xy^3z^4)_y = z + 6xy^2z^4$$
$$f_{yz}(x, y, z) = (xz + 3x^2y^2z^4)_z = x + 12x^2y^2z^3$$
$$f_{zy}(x, y, z) = (xy + 4x^2y^3z^3)_y = x + 12x^2y^2z^3$$

Finally, obtain the required higher mixed partials:

$$f_{xyz}(x, y, z) = (z + 6xy^2z^4)_z = 1 + 24xy^2z^3$$
$$f_{yzx}(x, y, z) = (x + 12x^2y^2z^3)_x = 1 + 24xy^2z^3$$
$$f_{zyx}(x, y, z) = (x + 12x^2y^2z^3)_x = 1 + 24xy^2z^3$$
■

11.3 | PROBLEM SET (Pg 52–75)

1–31, 47, 48, 49, 52–57

A 1. **WHAT DOES THIS SAY?** What is a partial derivative?

2. **Exploration Problem** Describe two fundamental interpretations of the partial derivatives $f_x(x, y)$ and $f_y(x, y)$.

Determine f_x, f_y, f_{xy}, and f_{yx} in Problems 3–8.

3. $f(x, y) = x^3 + x^2y + xy^2 + y^3$
4. $f(x, y) = (x + xy + y)^3$
5. $f(x, y) = \dfrac{x}{y}$
6. $f(x, y) = xe^{xy}$
7. $f(x, y) = \ln(2x + 3y)$
8. $f(x, y) = \sin x^2y$

Determine f_x and f_y in Problems 9–16.

9. **a.** $f(x, y) = (\sin x^2)\cos y$ **b.** $f(x, y) = \sin(x^2 \cos y)$
10. **a.** $f(x, y) = (\sin\sqrt{x})\ln y^2$ **b.** $f(x, y) = \sin(\sqrt{x}\ln y^2)$
11. $f(x, y) = \sqrt{3x^2 + y^4}$
12. $f(x, y) = xy^2\ln(x + y)$
13. $f(x, y) = x^2e^{x+y}\cos y$
14. $f(x, y) = xy^3\tan^{-1} y$
15. $f(x, y) = \sin^{-1}(xy)$
16. $f(x, y) = \cos^{-1}(xy)$

Determine f_x, f_y, and f_z in Problems 17–22.

17. $f(x, y, z) = xy^2 + yz^3 + xyz$
18. $f(x, y, z) = xye^z$
19. $f(x, y, z) = \dfrac{x + y^2}{z}$
20. $f(x, y, z) = \dfrac{xy + yz}{xz}$
21. $f(x, y, z) = \ln(x + y^2 + z^3)$
22. $f(x, y, z) = \sin(xy + z)$

In Problems 23–28, determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ by differentiating implicitly.

23. $\dfrac{x^2}{9} - \dfrac{y^2}{4} + \dfrac{z^2}{2} = 1$
24. $3x^2 + 4y^2 + 2z^2 = 5$
25. $3x^2y + y^3z - z^2 = 1$
26. $x^3 - xy^2 + yz^2 - z^3 = 4$
27. $\sqrt{x} + y^2 + \sin xz = 2$
28. $\ln(xy + yz + xz) = 5 \ (x > 0, \quad y > 0, \quad z > 0)$

In Problems 29–32, compute the slope of the tangent line to the graph of f at the given point P_0 in the direction parallel to

a. the xz-plane *b. the yz-plane*

29. $f(x, y) = xy^3 + x^3y$; $P_0(1, -1, -2)$
30. $f(x, y) = \dfrac{x^2 + y^2}{xy}$; $P_0(1, -1, -2)$
31. $f(x, y) = x^2 \sin(x + y)$; $P_0\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$
32. $f(x, y) = x \ln(x + y^2)$; $P_0(e, 0, e)$

B 33. Determine f_x and f_y for

$$f(x, y) = \int_x^y (t^2 + 2t + 1)\, dt$$

Hint: Review the second fundamental theorem of calculus.

34. Determine f_x and f_y for

$$f(x, y) = \int_{x^2}^{2y} (e^t + 3t)\, dt$$

Hint: Review the second fundamental theorem of calculus.

A function $f(x, y)$ is said to be ***harmonic*** *on the open set S iff f_{xx} and f_{yy} are continuous and*

$$f_{xx} + f_{yy} = 0$$

throughout S. Show that each function in Problems 35–38 is harmonic on the given set.

35. $f(x, y) = 3x^2y - y^3$; S is the entire plane.
36. $f(x, y) = \ln(x^2 + y^2)$; S is the plane with the point $(0, 0)$ removed.
37. $f(x, y) = e^x \sin y$; S is the entire plane.
38. $f(x, y) = \sin x \cosh y$; S is the entire plane.
39. For $f(x, y) = \cos xy^2$, show $f_{xy} = f_{yx}$.
40. For $f(x, y) = (\sin^2 x)(\sin y)$, show $f_{xy} = f_{yx}$.
41. Find $f_{xzy} - f_{yzz}$, where $f(x, y, z) = x^2 + y^2 - 2xy \cos z$.
42. Two commodities Q_1 and Q_2 are said to be *substitute commodities* if an increase in the demand for either results in a decrease in the demand of the other. Let $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$ be the demand functions for Q_1 and Q_2, respectively, where p_1 and p_2 are the respective unit prices for the commodities.
    a. Explain why $\dfrac{\partial D_1}{\partial p_1} < 0$ and $\dfrac{\partial D_2}{\partial p_2} < 0$.
    b. Are $\dfrac{\partial D_1}{\partial p_2}$ and $\dfrac{\partial D_2}{\partial p_1}$ positive or negative? Explain.
    c. Give examples of substitute commodities.
43. Two commodities Q_1 and Q_2 are said to be *complementary commodities* if a decrease in the demand for either results in a decrease in the demand of the other. Let $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$ be the demand functions for Q_1 and Q_2, respectively, where p_1 and p_2 are the respective unit prices for the commodities.
    a. Is it true that $\dfrac{\partial D_1}{\partial p_1} < 0$ and $\dfrac{\partial D_2}{\partial p_2} < 0$? Explain.
    b. Determine whether $\dfrac{\partial D_1}{\partial p_2}$ and $\dfrac{\partial D_2}{\partial p_1}$ are positive or negative? Explain.
    c. Give examples of complementary commodities.
44. **Modeling Problem** The flow (in cm^3/s) of blood from an artery into a small capillary can be modeled by

$$F(x, y, z) = \frac{c\pi x^2}{4}\sqrt{y - z}$$

for constant $c > 0$, where x is the diameter of the capillary, y is the pressure in the artery, and z is the pressure in the capillary. Compute the rate of change of the flow of blood with respect to
    a. the diameter of the capillary
    b. the arterial pressure
    c. the capillary pressure
45. **Modeling Problem** Biologists have studied the oxygen consumption of certain furry mammals. They have found that if the mammal's body temperature is T degrees Celsius, fur temperature is t degrees Celsius, and the mammal does not sweat, then its relative oxygen consumption can be modeled by

$$C(m, t, T) = \sigma(T - t)m^{-0.67}$$

(kg/h), where m is the mammal's mass (in kg) and σ is a physical constant. Compute the rate (rounded to two decimal places) at which the oxygen consumption changes with respect to
    a. the mass m
    b. the body temperature T
    c. the fur temperature t
46. **Modeling Problem** A gas that gathers on a surface in a condensed layer is said to be *adsorbed* on the surface, and the surface is called an *adsorbing surface*. The amount of gas adsorbed per unit area on an adsorbing surface can be modeled by

$$S(p, T, h) = ape^{h/(bT)}$$

where p is the gas pressure, T is the temperature of the gas, h is the heat of the adsorbed layer of gas, and a and b are physical constants. Compute the rate of change of S with respect to
    a. p b. h c. T
47. The *ideal gas law* says that $PV = kT$, where P is the pressure of a confined gas, V is the volume, T is the temperature, and k is a physical constant.

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a. Calculate $\dfrac{\partial V}{\partial T}$.

b. Calculate $\dfrac{\partial P}{\partial V}$.

c. Show that $\dfrac{\partial P}{\partial V} \cdot \dfrac{\partial V}{\partial T} \cdot \dfrac{\partial T}{\partial P} = -1$.

48. At a certain factory, the output is given by the production function $Q = 120K^{2/3}L^{2/5}$, where K denotes the capital investment (in units of \$1,000) and L measures the size of the labor force (in worker-hours).

a. Determine the *marginal productivity of capital*, $\partial Q/\partial K$, and the *marginal productivity of labor*, $\partial Q/\partial L$.

b. Determine the signs of the second-order partial derivatives $\partial^2 Q/\partial L^2$ and $\partial^2 Q/\partial K^2$, and give an economic interpretation.

49. The temperature at a point (x, y) on a given metal plate in the xy-plane is determined according to the formula $T(x, y) = x^3 + 2xy^2 + y$ degrees. Compute the rate at which the temperature changes with distance if we start at $(2, 1)$ and move

a. parallel to the vector $\mathbf{j}$

b. parallel to the vector $\mathbf{i}$

50. In physics, the *wave equation* is

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

and the *heat equation* is

$$\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}.$$

In each of the following cases, determine whether z satisfies the wave equation, the heat equation, or neither.

a. $z = e^{-t}\left(\sin \dfrac{x}{c} + \cos \dfrac{x}{c}\right)$

b. $z = \sin 3ct \sin 3x$

c. $z = \sin 5ct \cos 5x$

51. **The Cauchy-Riemann equations** are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $u = u(x, y)$ and $v = v(x, y)$. Which (if any) of the following functions satisfy the Cauchy-Riemann equations?

a. $u = e^{-x}\cos y$, $v = e^{-x}\sin y$

b. $u = x^2 + y^2$, $v = 2xy$

c. $u = \ln(x^2 + y^2)$, $v = 2\tan^{-1}\left(\dfrac{y}{x}\right)$

52. When two resistors R_1 and R_2 are connected in parallel, their combined resistance R satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

What is $\dfrac{\partial^2 R}{\partial R_1^2}\dfrac{\partial^2 R}{\partial R_2^2}$?

53. Show that the production function $P(L, K) = L^{\alpha}K^{\beta}$, where α and β are constants, satisfies

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P$$

54. The kinetic energy of a body of mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that

$$\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K$$

55. A study of ground penetration near a toxic waste dump models the amount of pollution P at a depth x (in feet) and time t (in years) by the function

$$P(x, t) = P_0 + P_1 e^{-kx}\sin(At - kx)$$

where A, k, P_0, and P_1 are constants. Show that $P(x, t)$ satisfies the diffusion equation

$$\frac{\partial P}{\partial t} = c^2\frac{\partial^2 P}{\partial x^2}$$

for a certain constant c.

56. **Counterexample Problem** Let

$$f(x, y) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that $f_x(0, y) = -y$ and $f_y(x, 0) = x$, for all x and y. Then show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$. Why does this not violate the equality of mixed partials theorem?

57. Show that $f_x(0, 0) = 0$ but $f_y(0, 0)$ does not exist, where

$$f(x, y) = \begin{cases} (x^2 + y)\sin\left(\dfrac{1}{x^2 + y^2}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

58. **Modeling Problem** Suppose a substance is injected into a tube containing a liquid solvent and that the tube is placed so that its axis is parallel to the x-axis, as shown in Figure 11.19.

Figure 11.19 Problem 58

Assume that the concentration of the substance varies only in the x-direction, and let $C(x, t)$ denote the concentration at position x and time t. Because the number of molecules in this substance is very large, it is reasonable to assume that C is a continuous function whose partial derivatives exist. One model for the flow yields the *diffusion equation in one dimension*, namely,

$$\frac{\partial C}{\partial t} = \delta\frac{\partial^2 C}{\partial x^2}$$

where δ is the *diffusion constant*.

a. What must δ be for a function of the form

$$C(x, t) = e^{ax+bt}$$

(a and b are constants) to satisfy the diffusion equation?

b. Verify that

$$C(x, t) = t^{-1/2}e^{-x^2/(4\delta t)}$$

satisfies the diffusion equation.

59. The area of a triangle is $A = \frac{1}{2}ab \sin \gamma$, where γ is the angle between sides of length a and b.

a. Determine $\dfrac{\partial A}{\partial a}$, $\dfrac{\partial A}{\partial b}$, and $\dfrac{\partial A}{\partial \gamma}$.

b. Suppose a is given as a function of b, A, and γ. What is $\dfrac{\partial a}{\partial \gamma}$?

60. Journal Problem *Crux*, problem by John A. Winterink* ■ Prove the validity of the following simple method for finding the center of a conic: For the central conic,

$$\phi(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$ab - h^2 \neq 0$, show that the center is the intersection of the lines $\partial\phi/\partial x = 0$ and $\partial\phi/\partial y = 0$.

*Problem 54, Vol. 6 (1980), p. 154.

Equation c

11.4 Tangent Planes, Approximations, and Differentiability

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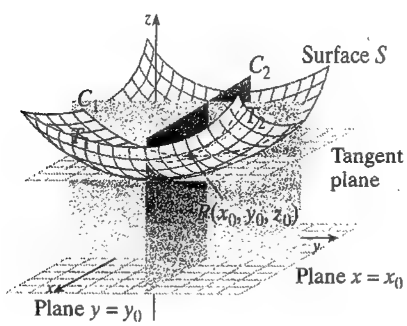
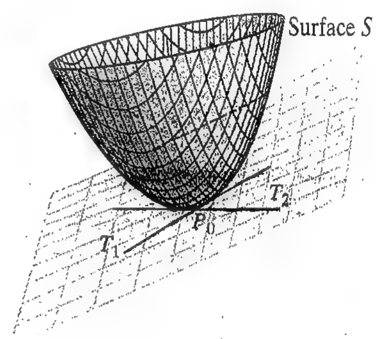
IN THIS SECTION tangent planes, incremental approximations, the total differential, differentiability

TANGENT PLANES

Suppose S is a surface with the equation $z = f(x, y)$, where f has continuous first partial derivatives f_x and f_y. Let $P_0(x_0, y_0, z_0)$ be a point on S, and let C_1 be the curve of intersection of S with the plane $x = x_0$ and C_2, the intersection of S with the plane $y = y_0$, as shown in Figure 11.20a. The tangent lines T_1 and T_2 to C_1 and C_2, respectively, at P_0 determine a unique plane, and in Section 11.6 we will find that this plane actually contains the tangent to every smooth curve on S that passes through P_0. We call this plane the **tangent plane** to S at P_0 (Figure 11.20b).



a. C_1 and C_2 are the curves of intersection of the surface S with the planes $x = x_0$ and $y = y_0$.



b. The tangent plane to S at P_0 contains the tangent lines T_1 and T_2 to C_1 and C_2, respectively.

Figure 11.20 Tangent plane to the surface S at the point P_0

To find an equation for the tangent plane, recall that the equation of a plane with normal $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

If $C \neq 0$, divide both sides by C and let $a = -A/C$ and $b = -B/C$ to obtain

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

The intersection of this plane and the plane $x = x_0$ is the tangent line T_1, which we know has slope $f_y(x_0, y_0)$ from the geometric interpretation of partial derivatives. Setting $x = x_0$ in the equation for the tangent plane, we find that T_1 has the point-slope form

$$z - z_0 = b(y - y_0)$$

so we must have $b = f_y(x_0, y_0) = \left.\dfrac{\partial z}{\partial y}\right|_{(x_0, y_0)}$. Similarly, setting $y = y_0$, we obtain

$$z - z_0 = a(x - x_0)$$

which represents the tangent line T_2, with slope $a = f_x(x_0, y_0)$. To summarize:

Equation of the Tangent Plane

> Suppose S is a surface with the equation $z = f(x, y)$ and let $P_0(x_0, y_0, z_0)$ be a point on S at which a tangent plane exists. Then an **equation for the tangent plane** to S at P_0 is
>
> $$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
>
> If the equation is written in the form
>
> $$Ax + By + Cz + D = 0$$
>
> we say the equation of the plane is in standard form.

EXAMPLE 1 Equation of a tangent plane for a surface defined by $z = f(x, y)$

Find an equation for the tangent plane to the surface $z = \tan^{-1} \dfrac{y}{x}$ at the point $P_0\left(1, \sqrt{3}, \frac{\pi}{3}\right)$.

Solution

$$f_x(x, y) = \frac{-yx^{-2}}{1 + \left(\frac{y}{x}\right)^2} = \frac{-y}{x^2 + y^2}; \qquad f_x\left(1, \sqrt{3}\right) = \frac{-\sqrt{3}}{1 + 3} = \frac{-\sqrt{3}}{4}$$

$$f_y(x, y) = \frac{x^{-1}}{1 + \left(\frac{y}{x}\right)^2} = \frac{x}{x^2 + y^2}; \qquad f_y\left(1, \sqrt{3}\right) = \frac{1}{1 + 3} = \frac{1}{4}$$

The equation of the tangent plane is

$$z - \frac{\pi}{3} = \left(\frac{-\sqrt{3}}{4}\right)(x - 1) + \frac{1}{4}\left(y - \sqrt{3}\right).$$

or, in standard form,

$$3\sqrt{3}x - 3y + 12z - 4\pi = 0$$

■

INCREMENTAL APPROXIMATIONS

In Chapter 3, we observed that the tangent line to the curve $y = f(x)$ at the point $P(x_0, y_0)$ is the line that best fits the shape of the curve in the immediate vicinity of P. That is, if f is differentiable at $x = x_0$ and the increment Δx is sufficiently small, then

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x$$

or, equivalently,

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

Similarly, the tangent plane at $P(x_0, y_0, z_0)$ is the plane that best fits the shape of the surface $z = f(x, y)$ near P, and the analogous **incremental** (or **linear**) approximation formula is as follows.

Incremental Approximation of a Function of Two Variables

If $f(x, y)$ and its partial derivatives f_x and f_y are defined in an open region R containing the point $P(x_0, y_0)$ and f_x and f_y are continuous at P, then

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

so that

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

A graphical interpretation of this incremental approximation formula is shown in Figure 11.21.

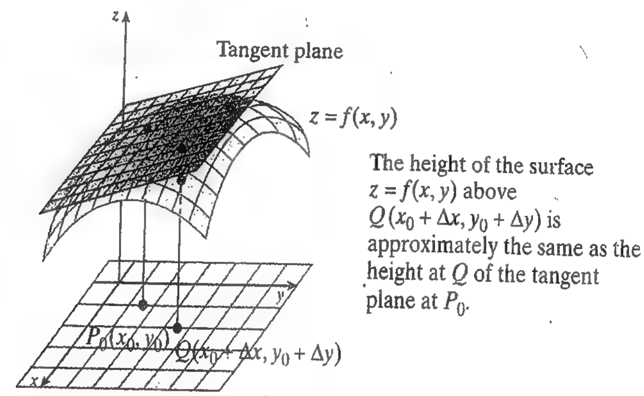


Figure 11.21 Incremental approximation to a function of two variables

The tangent plane to the surface $z = f(x, y)$ has the equation

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

or

$$z - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

As long as we are near (x_0, y_0), the height of the tangent plane is approximately the same as the height of the surface. Thus, if $|\Delta x|$ and $|\Delta y|$ are small, the point $(x_0 + \Delta x, y_0 + \Delta y)$ will be near (x_0, y_0) and we have

$$\underbrace{f(x_0 + \Delta x, y_0 + \Delta y)}_{\substack{\text{Height of } z = f(x,y) \\ \text{above } Q(x_0 + \Delta x, y_0 + \Delta y)}} \approx \underbrace{f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y}_{\text{Height of the tangent plane above } Q}$$

Increments of a function of three variables $f(x, y, z)$ can be defined in a similar fashion. Suppose f has continuous partial derivatives f_x, f_y, f_z in a ball centered at the point (x_0, y_0, z_0). Then if the numbers Δx, Δy, Δz are all sufficiently small, we have

$$\begin{aligned}\Delta f &= f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0) \\ &\approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z\end{aligned}$$

EXAMPLE 2 Using increments to estimate the change of a function

An open box has length 3 ft, width 1 ft, and height 2 ft and is constructed from material that costs $\$2/\text{ft}^2$ for the sides and $\$3/\text{ft}^2$ for the bottom (see Figure 11.22). Compute the cost of constructing the box, and then use increments to estimate the

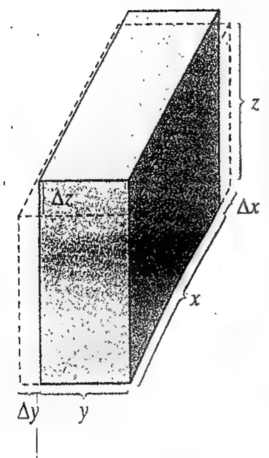


Figure 11.22 Construction of a box

change in cost if the length and width are each increased by 3 in. and the height is decreased by 4 in.

Solution

An open (no top) box with length x, width y, and height z has surface area

$$S = \underbrace{xy}_{\text{Bottom}} + \underbrace{2xz + 2yz}_{\text{Four side faces}}$$

Because the sides cost $\$2/\text{ft}^2$ and the bottom $\$3/\text{ft}^2$, the total cost is

$$C(x, y, z) = 3xy + 2(2xz + 2yz)$$

The partial derivatives of C are

$$C_x = 3y + 4z \qquad C_y = 3x + 4z \qquad C_z = 4x + 4y$$

and the dimensions of the box change by

$$\Delta x = \tfrac{3}{12} = 0.25 \text{ ft} \qquad \Delta y = \tfrac{3}{12} = 0.25 \text{ ft} \qquad \Delta z = \tfrac{-4}{12} \approx -0.33 \text{ ft}$$

Thus, the change in the total cost is approximated by

$$\begin{aligned}\Delta C &\approx C_x(3,1,2)\Delta x + C_y(3,1,2)\Delta y + C_z(3,1,2)\Delta z\\ &= [3(1) + 4(2)](0.25) + [3(3) + 4(2)](0.25) + [4(3) + 4(1)]\left(-\tfrac{4}{12}\right)\\ &\approx 1.67\end{aligned}$$

That is, the cost increases by approximately \$1.67. ■

EXAMPLE 3 Maximum percentage error using differentials

The radius and height of a right circular cone are measured with errors of at most 3% and 2%, respectively. Use increments to approximate the maximum possible percentage error in computing the volume of the cone using these measurements and the formula $V = \frac{1}{3}\pi R^2 H$.

Solution

We are given that

$$\left|\frac{\Delta R}{R}\right| \le 0.03 \quad \text{and} \quad \left|\frac{\Delta H}{H}\right| \le 0.02$$

The partial derivatives of V are

$$V_R = \tfrac{2}{3}\pi RH \quad \text{and} \quad V_H = \tfrac{1}{3}\pi R^2$$

so the change in V is approximated by

$$\Delta V \approx \left(\tfrac{2}{3}\pi RH\right)\Delta R + \left(\tfrac{1}{3}\pi R^2\right)\Delta H$$

Dividing by the volume $V = \frac{1}{3}\pi R^2 H$, we obtain

$$\frac{\Delta V}{V} \approx \frac{\frac{2}{3}\pi RH\Delta R + \frac{1}{3}\pi R^2 \Delta H}{\frac{1}{3}\pi R^2 H} = 2\left(\frac{\Delta R}{R}\right) + \left(\frac{\Delta H}{H}\right)$$

so that $\left|\dfrac{\Delta V}{V}\right| \le 2\left|\dfrac{\Delta R}{R}\right| + \left|\dfrac{\Delta H}{H}\right| \le 2(0.03) + (0.02) = 0.08$. Thus, the maximum percentage error in computing the volume V is approximately 8%. ■

THE TOTAL DIFFERENTIAL

For a function of one variable, $y = f(x)$, we defined the differential dy to be $dy = f'(x)dx$. For the two-variable case, we make the following analogous definition.

Total Differential

The **total differential** of the function $f(x, y)$ is

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = f_x(x, y)\,dx + f_y(x, y)\,dy$$

where dx and dy are independent variables. Similarly, for a function of three variables $w = f(x, y, z)$ the **total differential** is

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

EXAMPLE 4 Total differential

Determine the total differential of the given functions:

a. $f(x, y, z) = 2x^3 + 5y^4 - 6z$ **b.** $f(x, y) = x^2 \ln(3y^2 - 2x)$

Solution

a. $$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 6x^2\,dx + 20y^3\,dy - 6\,dz$$

b. $$\begin{aligned} df &= \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \\ &= \left[2x\ln(3y^2 - 2x) + x^2\frac{-2}{3y^2 - 2x}\right]dx + \left[x^2\frac{6y}{3y^2 - 2x}\right]dy \\ &= \left[2x\ln(3y^2 - 2x) - \frac{2x^2}{3y^2 - 2x}\right]dx + \frac{6x^2y}{3y^2 - 2x}\,dy \end{aligned}$$

EXAMPLE 5 Application of the total differential

At a certain factory, the daily output is $Q = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment (in units of \$1,000) and L the size of the labor force (in worker-hours). The current capital investment is \$900,000, and 1,000 worker-hours of labor are used each day. Estimate the change in output that will result if capital investment is increased by \$1,000 and labor is decreased by 2 worker-hours.

Solution

The change in output is estimated by the total differential dQ. We have $K = 900$, $L = 1{,}000$, $dK = \Delta K = 1$, and $dL = \Delta L = -2$. The total differential of $Q(x, y)$ is

$$\begin{aligned} dQ &= \frac{\partial Q}{\partial K}\,dK + \frac{\partial Q}{\partial L}\,dL \\ &= 60\left(\tfrac{1}{2}\right)K^{-1/2}L^{1/3}\,dK + 60\left(\tfrac{1}{3}\right)K^{1/2}L^{-2/3}\,dL \\ &= 30K^{-1/2}L^{1/3}\,dK + 20K^{1/2}L^{-2/3}\,dL \end{aligned}$$

Substituting for K, L, dK, and dL,

$$dQ = 30(900)^{-1/2}(1{,}000)^{1/3}(1) + 20(900)^{1/2}(1{,}000)^{-2/3}(-2) = -2$$

Thus, the output decreases by approximately 2 units when the capital investment is increased by \$1,000 and labor is decreased by 2 worker-hours. ■

EXAMPLE 6 Maximum percentage error in an electrical circuit

When two resistances R_1 and R_2 are connected in parallel, the total resistance R satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 is measured as 300 ohms with a maximum error of 2% and R_2 is measured as 500 ohms with a maximum error of 3%, what is the maximum percentage error in R?

Solution

We are given that

$$\left|\frac{dR_1}{R_1}\right| \leq 0.02 \quad \text{and} \quad \left|\frac{dR_2}{R_2}\right| \leq 0.03$$

and we wish to find the maximum value of $\left|\frac{dR}{R}\right|$. Because $R = \frac{R_1R_2}{R_1 + R_2}$ (solve the given equation for R), we have

$$\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1 + R_2)^2} \quad \text{and} \quad \frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1 + R_2)^2} \qquad \textit{Quotient rule}$$

It follows that the total differential of R is

$$\begin{aligned} dR &= \frac{\partial R}{\partial R_1}dR_1 + \frac{\partial R}{\partial R_2}dR_2 \\ &= \frac{R_2^2}{(R_1 + R_2)^2}dR_1 + \frac{R_1^2}{(R_1 + R_2)^2}dR_2 \end{aligned}$$

We now find $\frac{dR}{R}$ by dividing both sides by R; however, since $\frac{1}{R} = \frac{R_1 + R_2}{R_1R_2}$, it follows that

$$\begin{aligned} dR \cdot \frac{1}{R} &= \left[\frac{R_2^2}{(R_1 + R_2)^2}dR_1 + \frac{R_1^2}{(R_1 + R_2)^2}dR_2\right] \cdot \frac{R_1 + R_2}{R_1R_2} \\ \frac{dR}{R} &= \frac{R_2}{R_1 + R_2} \cdot \frac{dR_1}{R_1} + \frac{R_1}{R_1 + R_2} \cdot \frac{dR_2}{R_2} \end{aligned}$$

Finally, apply the triangle inequality (Table 1.1) to this relationship:

$$\begin{aligned} \left|\frac{dR}{R}\right| &\leq \left|\frac{R_2}{R_1 + R_2}\right|\left|\frac{dR_1}{R_1}\right| + \left|\frac{R_1}{R_1 + R_2}\right|\left|\frac{dR_2}{R_2}\right| \\ &\leq \frac{500}{300 + 500}(0.02) + \frac{300}{300 + 500}(0.03) = 0.02375 \end{aligned}$$

The maximum percentage is approximately 2.4%. ■

DIFFERENTIABILITY

Recall from Chapter 3 that if $f(x)$ is differentiable at x_0, its increment is

$$\Delta f = f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x + \epsilon \Delta x$$

where $\epsilon \to 0$ as $\Delta x \to 0$ (see Figure 11.23).

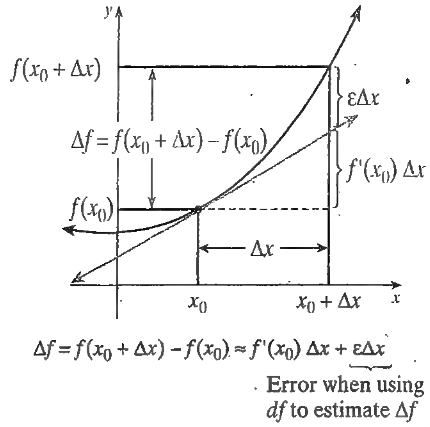


Figure 11.23 Increment of a function f

For a function of two variables, the increment of x is an independent variable denoted by Δx, the increment of y is an independent variable denoted by Δy, and the increment of f at (x_0, y_0) is defined as

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

We use this increment representation to define differentiability as follows:

Definition of Differentiability

The function $f(x, y)$ is **differentiable** at (x_0, y_0) if the increment of f can be expressed as

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $\Delta x \to 0$ and $\Delta y \to 0$. In addition, $f(x, y)$ is said to be **differentiable in the region R** of the plane if f is differentiable at each point in R.

In Section 3.1, we showed that a function of one variable is continuous wherever it is differentiable. The following theorem establishes the same result for a function of two variables.

THEOREM 11.2 **Differentiability implies continuity**

If $f(x, y)$ is differentiable at (x_0, y_0), it is also continuous there.

Proof We wish to show that $f(x, y) \to f(x_0, y_0)$ as $(x, y) \to (x_0, y_0)$ or, equivalently, that

$$\lim_{(x,y)\to(x_0,y_0)} [f(x, y) - f(x_0, y_0)] = 0$$

WARNING Example ; Theorem centered continuou

If we set $\Delta x = x - x_0$ and $\Delta y = y - y_0$ and let Δf denote the increment of f at (x_0, y_0), we have (by substitution)

$$f(x, y) - f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = \Delta f$$

Then, because $(\Delta x, \Delta y) \to (0, 0)$ as $(x, y) \to (x_0, y_0)$, we wish to prove that

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \Delta f = 0$$

Since f is differentiable at (x_0, y_0), we have

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. It follows that

$$\begin{aligned}\lim_{(\Delta x,\Delta y)\to(0,0)} \Delta f &= \lim_{(\Delta x,\Delta y)\to(0,0)} [f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y] \\ &= [f_x(x_0, y_0)] \cdot 0 + [f_y(x_0, y_0)] \cdot 0 + 0 + 0 = 0\end{aligned}$$

as required. ❑

WARNING Be careful about how you use the word *differentiable.* In the single-variable case, a function is differentiable at a point if its derivative exists there. However, the word is used differently for a function of two variables. In particular, the existence of the partial derivatives f_x and f_y does not guarantee that the function is differentiable, as illustrated in the following example.

EXAMPLE 7 A nondifferentiable function for which f_x and f_y exist

Let

$$f(x, y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

That is, the function f has the value 1 when (x, y) is in the first quadrant and is 0 elsewhere. Show that the partial derivatives f_x and f_y exist at the origin, but f is not differentiable there.

Solution

Since $f(0, 0) = 0$, we have

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = 0$$

and similarly, $f_y(0, 0) = 0$. Thus, the partial derivatives both exist at the origin.

If $f(x, y)$ were differentiable at the origin, it would have to be continuous there (Theorem 11.2). Thus, we can show f is not differentiable by showing that it is not continuous at (0, 0). Toward this end, note that $\lim_{(x,y)\to(0,0)} f(x, y)$ is 1 along the line $y = x$ in the first quadrant but is 0 if the approach is along the x-axis. This means that the limit does not exist. Thus, f is not continuous at (0, 0) and consequently is also not differentiable there. ■

Although the existence of partial derivatives at $P(x_0, y_0)$ is not enough to guarantee that $f(x, y)$ is differentiable at P, we do have the following sufficient condition for differentiability.

WARNING Note that the function in Example 7 does not contradict Theorem 11.3 because there is no disk centered at (0, 0) on which f is continuous.

THEOREM 11.3 Sufficient condition for differentiability

If f is a function of x and y, and f, f_x, and f_y are continuous in a disk D centered at (x_0, y_0), then f is differentiable at (x_0, y_0).

Proof The proof is found in advanced calculus texts. ❑

EXAMPLE 8 Establishing differentiability

Show that $f(x, y) = x^2y + xy^3$ is differentiable for all (x, y).

Solution

Compute the partial derivatives

$$f_x(x, y) = \frac{\partial}{\partial x}(x^2y + xy^3) = 2xy + y^3$$

$$f_y(x, y) = \frac{\partial}{\partial y}(x^2y + xy^3) = x^2 + 3xy^2$$

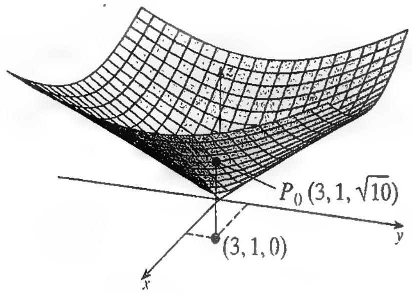
Because f, f_x, and f_y are all polynomials in x and y, they are continuous throughout the plane. Therefore, the sufficient condition for differentiability theorem assures us that f must be differentiable for all x and y. ■

Guidelines 8.1–35, 41, 43, 49

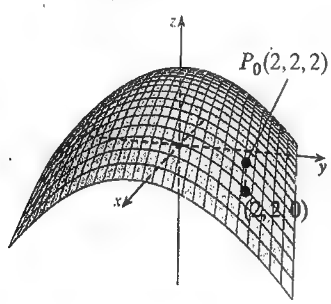
11.4 PROBLEM SET (Pg 79–94 §51–34) (Pg

A *In Problems 1–6, determine the standard-form equations for the tangent plane to the given surface at the prescribed point P_0.*

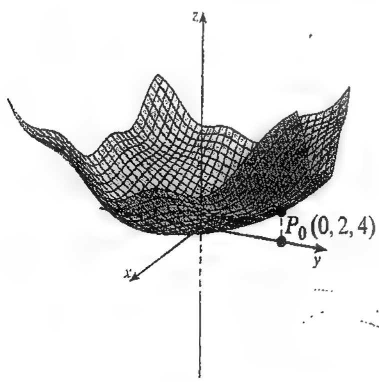
1. $z = \sqrt{x^2 + y^2}$ at $P_0(3, 1, \sqrt{10})$



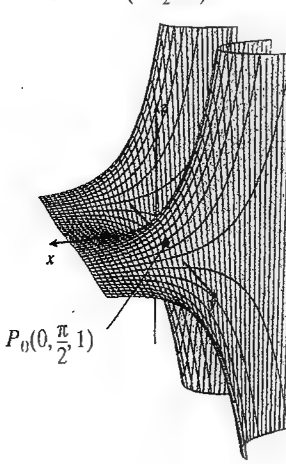
2. $z = 10 - x^2 - y^2$ at $P_0(2, 2, 2)$



3. $f(x, y) = x^2 + y^2 + \sin xy$ at $P_0 = (0, 2, 4)$



4. $f(x, y) = e^{-x} \sin y$ at $P_0\left(0, \frac{\pi}{2}, 1\right)$



5. $z = \tan^{-1} \dfrac{y}{x}$ at $P_0\left(2, 2, \frac{\pi}{4}\right)$
6. $z = \ln |x + y^2|$ at $P_0(-3, -2, 0)$

Determine the total differential of the functions given in Problems 7–18.

7. $f(x, y) = 5x^2y^3$
8. $f(x, y) = 8x^3y^2 - x^4y^5$
9. $f(x, y) = \sin xy$
10. $f(x, y) = \cos x^2y$
11. $f(x, y) = \dfrac{y}{x}$
12. $f(x, y) = \dfrac{x^2}{y}$
13. $f(x, y) = ye^x$
14. $f(x, y) = e^{x^2+y}$
15. $f(x, y, z) = 3x^3 - 2y^4 + 5z$
16. $f(x, y, z) = \sin x + \sin y + \cos z$
17. $f(x, y, z) = z^2 \sin(2x - 3y)$
18. $f(x, y, z) = 3y^2z \cos x$

Show that the functions in Problems 19–22 are differentiable for all (x, y).

19. $f(x, y) = xy^3 + 3xy^2$
20. $f(x, y) = x^2 + 4x - y^2$
21. $f(x, y) = e^{2x+y^2}$
22. $f(x, y) = \sin(x^2 + 3y)$

Use an incremental approximation to estimate the functions at the values given in Problems 23–28. Check by using a calculator.

23. $f(1.01, 2.03)$, where $f(x, y) = 3x^4 + 2y^4$
24. $f(0.98, 1.03)$, where $f(x, y) = x^5 - 2y^3$
25. $f\left(\frac{\pi}{2} + 0.01, \frac{\pi}{2} - 0.01\right)$, where $f(x, y) = \sin(x + y)$
26. $f\left(\sqrt{\frac{\pi}{2}} + 0.01, \sqrt{\frac{\pi}{2}} - 0.01\right)$, where $f(x, y) = \sin(xy)$
27. $f(1.01, 0.98)$, where $f(x, y) = e^{xy}$
28. $f(1.01, 0.98)$, where $f(x, y) = e^{x^2y^2}$

B 29. Find an equation for each horizontal tangent plane to the surface
$$z = 5 - x^2 - y^2 + 4y$$

30. Find an equation for each horizontal tangent plane to the surface
$$z = 4(x - 1)^2 + 3(y + 1)^2$$

31. **a.** Show that if x and y are sufficiently close to zero and f is differentiable at $(0, 0)$, then
$$f(x, y) \approx f(0, 0) + xf_x(0, 0) + yf_y(0, 0).$$
b. Use the approximation formula in part **a** to show that
$$\frac{1}{1 + x - y} \approx 1 - x + y$$
for small x and y.
c. If x and y are sufficiently close to zero, what is the approximate value of the expression
$$\frac{1}{(x + 1)^2 + (y + 1)^2}$$

32. When two resistors with resistances P and Q ohms are connected in parallel, the combined resistance is R, where
$$\frac{1}{R} = \frac{1}{P} + \frac{1}{Q}$$

If P and Q are measured at 6 and 10 ohms, respectively, with errors no greater than 1%, what is the maximum percentage error in the computation of R?

33. A closed box is found to have length 2 ft, width 4 ft, and height 3 ft, where the measurement of each dimension is made with a maximum possible error of ± 0.02 ft. The top of the box is made from material that costs $\$2/\text{ft}^2$; and the material for the sides and bottom costs only $\$1.50/\text{ft}^2$. What is the maximum error involved in the computation of the cost of the box?

34. A cylindrical tank is 4 ft high and has an outer diameter of 2 ft. The walls of the tank are 0.2 in. thick. Approximate the volume of the interior of the tank assuming the tank has a top and a bottom that are both also 0.2 in. thick.

35. The Higrade Company sells two brands, X and Y, of a commercial soap, in thousand-pound units. If x units of brand X and y units of brand Y are sold, the unit price for brand X is

$$p(x) = 4{,}000 - 500x$$

and that of brand Y is

$$q(y) = 3{,}000 - 450y$$

a. Find an expression for the total revenue of R in terms of p and q.

b. Suppose brand X sells for \$500 per unit and brand Y sells for \$750 per unit. Estimate the change in total revenue if the unit prices are increased by \$20 for brand X and \$18 for brand Y.

36. The output at a certain factory is

$$Q = 150\, K^{2/3}\, L^{1/3}$$

where K is the capital investment in units of \$1,000 and L is the size of the labor force, measured in worker-hours. The current capital investment is \$500,000 and \$1,500 worker-hours of labor are used.

a. Estimate the change in output that results when capital investment is increased by \$700 and labor is increased by 6 worker-hours.

b. What if capital investment is increased by \$500 and labor is decreased by 4 worker-hours?

37. According to Poiseuille's law, the resistance to the flow of blood offered by a cylindrical blood vessel of radius r and length x is

$$R(r, x) = \frac{cx}{r^4}$$

for a constant $c > 0$. A certain blood vessel in the body is 8 cm long and has a radius of 2 mm. Estimate the percentage change in R when x is increased by 3% and r is decreased by 2%.

38. For 1 mole of an ideal gas, the volume V, pressure P, and absolute temperature T are related by the equation $PV = kT$, where k is a certain fixed constant that depends on the gas. Suppose we know that if $T = 400$ (absolute) and $P = 3{,}000\ \text{lb/ft}^2$, then $V = 14\ \text{ft}^3$. Approximate the change in pressure if the temperature and volume are increased to 403 and $14.1\ \text{ft}^3$, respectively.

39. **Modeling Problem** If x gram-moles of sulfuric acid are mixed with y gram-moles of water, the heat liberated is modeled by

$$F(x, y) = \frac{1.786xy}{1.798x + y} \text{ cal}$$

Approximately how much additional heat is generated if a mixture of 5 gram-moles of acid and 4 gram-moles of water is increased to a mixture of 5.1 gram-moles of acid and 4.04 gram-moles of water?

40. **Modeling Problem** A business analyst models the sales of a new product by the function

$$Q(x, y) = 20x^{3/2}\, y$$

where x thousand dollars are spent on development and y thousand dollars on promotion. Current plans call for the expenditure of \$36,000 on development and \$25,000 on promotion. Use the total differential of Q to estimate the change in sales that will result if the amount spent on development is increased by \$500 and the amount spent on promotion is decreased by \$500.

41. Using x hours of skilled labor and y hours of unskilled labor, a manufacturer can produce $f(x, y) = 10xy^{1/2}$ units. Currently, the manufacturer has used 30 h of skilled labor and 36 h of unskilled labor and is planning to use 1 additional hour of skilled labor. Use calculus to estimate the corresponding change that the manufacturer should make in the level of unskilled labor so that the total output will remain the same.

42. **Modeling Problem** A grocer's weekly profit from the sale of two brands of orange juice is modeled by

$$P(x, y) = (x - 30)(70 - 5x + 4y) + (y - 40)(80 + 6x - 7y)$$

dollars, where x cents is the price per can of the first brand and y cents is the price per can of the second. Currently, the first brand sells for 50¢ per can and the second for 52¢ per can. Use the total differential to estimate the change in the weekly profit that will result if the grocer raises the price of the first brand by 1¢ per can and lowers the price of the second brand by 2¢ per can.

43. A juice can is 12 cm tall and has a radius of 3 cm. A manufacturer is planning to reduce the height of the can by 0.2 cm and the radius by 0.3 cm. Use a total differential to estimate the percentage decrease in volume that occurs when the new cans are introduced. (Round to the nearest percent.)

44. **Modeling Problem** It is known that the period T of a simple pendulum with small oscillations is modeled by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum and g is the acceleration due to gravity: For a certain pendulum, it is known that $L = 4.03$ ft. It is also known that $g = 32.2\ \text{ft/s}^2$. What is the approximate error in calculating T by using $L = 4$ and $g = 32$?

45. If the weight of an object that does not float in water is x pounds in the air and its weight in water is y pounds, then the specific gravity of the object is

$$S = \frac{x}{x - y}$$

For a certain object, x and y are measured to be 1.2 lb and 0.5 lb, respectively. It is known that the measuring instrument will not register less than the true weights, but it could register more than the true weights by as much as 0.01 lb. What is the maximum possible error in the computation of the specific gravity?

46. A football has the shape of the ellipsoid

$$\frac{x^2}{9}+\frac{y^2}{36}+\frac{z^2}{9}=1$$

where the dimensions are in inches, and is made of leather 1/8 inch thick. Use differentials to estimate the volume of the leather shell. *Hint*: The ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

has volume $V=\frac{4}{3}\pi abc$.

47. Show that the following function is not differentiable at (0, 0):

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

48. Compute the total differentials

$$d\left(\frac{x}{x-y}\right) \quad \text{and} \quad d\left(\frac{y}{x-y}\right)$$

Why are these differentials equal?

49. Let A be the area of a triangle with sides a and b separated by an angle θ, as shown in Figure 11.24,

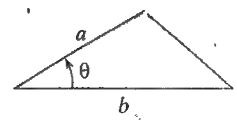


Figure 11.24 Problem 49

Suppose $\theta=\frac{\pi}{6}$, and a is increased by 4% while b is decreased by 3%. Use differentials to estimate the percentage change in A.

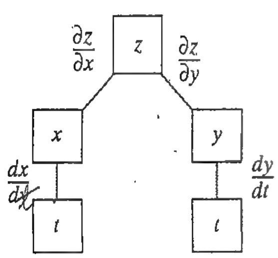
50. In Problem 49, suppose that θ also changes by no more than 2%. What is the maximum percentage change in A?

11.5 Chain Rules

IN THIS SECTION chain rule for one parameter, extensions of the chain rule

CHAIN RULE FOR ONE PARAMETER

We begin with a differentiable function of two variables $f(x, y)$. If $x = x(t)$ and $y = y(t)$ are, in turn, functions of a single parameter t, then $z = f(x(t), y(t))$ is a composite function of a parameter t. In this case, the chain rule for finding the derivative with respect to one parameter can now be stated.



Tree diagram illustrating the chain rule

***THEOREM 11.4* The chain rule for one independent parameter**

Let $f(x, y)$ be a differentiable function of x and y, and let $x = x(t)$ and $y = y(t)$ be differentiable functions of t. Then $z = f(x, y)$ is a differentiable function of t, and

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

➡ **What This Says** The tree diagram shown in the margin is a device for remembering the chain rule. The diagram begins at the top with the dependent variable z and cascades downward in two branches, first to the independent variables x and y, and then to the parameter t on which they each depend. Each branch segment is labeled with a derivative, and the chain rule is obtained by first multiplying the derivatives on the two segments of each branch and then adding to obtain

$$\frac{dz}{dt}=\underbrace{\frac{\partial z}{\partial x}\frac{dx}{dt}}_{\textit{left branch}}+\underbrace{\frac{\partial z}{\partial y}\frac{dy}{dt}}_{\textit{right branch}}$$

Proof Recall that because $z = f(x, y)$ is differentiable, we can write the increment Δz in the following form:

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $\Delta x \to 0$ and $\Delta y \to 0$. Dividing by $\Delta t \neq 0$, we obtain

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}$$

Because x and y are functions of t, we can write their increments as

$$\Delta x = x(t + \Delta t) - x(t) \quad \text{and} \quad \Delta y = y(t + \Delta t) - y(t)$$

We know that x and y both vary continuously with t (remember, they are differentiable), and it follows that $\Delta x \to 0$ and $\Delta y \to 0$ as $\Delta t \to 0$, so that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta t \to 0$. Therefore, we have

$$\begin{aligned}\frac{dz}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0}\left[\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}\right] \\ &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + 0\frac{dx}{dt} + 0\frac{dy}{dt}\end{aligned}$$

□

EXAMPLE 1 Verifying the chain rule explicitly

Let $z = x^2 + y^2$, where $x = \dfrac{1}{t}$ and $y = t^2$. Compute $\dfrac{dz}{dt}$ in two ways:

a. by first expressing z explicitly in terms of t
b. by using the chain rule

Solution

a. By substituting $x = \dfrac{1}{t}$ and $y = t^2$, we find that

$$z = x^2 + y^2 = \left(\frac{1}{t}\right)^2 + (t^2)^2 = t^{-2} + t^4 \qquad \text{for} \quad t \neq 0$$

Thus, $\dfrac{dz}{dt} = -2t^{-3} + 4t^3$.

b. Because $z = x^2 + y^2$ and $x = t^{-1}$, $y = t^2$,

$$\frac{\partial z}{\partial x} = 2x; \qquad \frac{\partial z}{\partial y} = 2y; \qquad \frac{dx}{dt} = -t^{-2}; \qquad \frac{dy}{dt} = 2t$$

Use the chain rule for one independent parameter:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2x)(-t^{-2}) + 2y(2t) && \textbf{Chain rule} \\ &= 2(t^{-1})(-t^{-2}) + 2(t^2)(2t) && \textbf{Substitute} \\ &= -2t^{-3} + 4t^3\end{aligned}$$

EXAMPLE 2 Chain rule for one independent parameter

Let $z = \sqrt{x^2 + 2xy}$, where $x = \cos\theta$ and $y = \sin\theta$. Find $\dfrac{dz}{d\theta}$.

Solution

$$\frac{\partial z}{\partial x} = \tfrac{1}{2}(x^2 + 2xy)^{-1/2}(2x + 2y) \quad \text{and} \quad \frac{\partial z}{\partial y} = \tfrac{1}{2}(x^2 + 2xy)^{-1/2}(2x)$$

Also, $\dfrac{dx}{d\theta} = -\sin\theta$ and $\dfrac{dy}{d\theta} = \cos\theta$. Use the chain rule for one independent parameter to find

$$\begin{aligned}\frac{dz}{d\theta} &= \frac{\partial z}{\partial x}\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\frac{dy}{d\theta} \\ &= \tfrac{1}{2}(x^2 + 2xy)^{-1/2}(2x + 2y)(-\sin\theta) + \tfrac{1}{2}(x^2 + 2xy)^{-1/2}(2x)(\cos\theta) \\ &= (x^2 + 2xy)^{-1/2}(x\cos\theta - x\sin\theta - y\sin\theta)\end{aligned}$$ ∎

EXAMPLE 3 Related rate application using the chain rule

A right circular cylinder (see Figure 11.25) is changing in such a way that its radius r is increasing at the rate of 3 in./min and its height h is decreasing at the rate of 5 in./min. At what rate is the volume of the cylinder changing when the radius is 10 in. and the height is 8 in.?

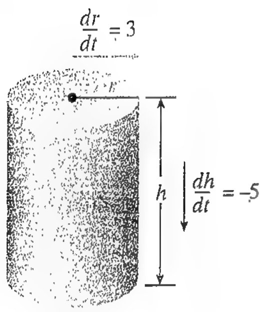


Figure 11.25 Right circular cylinder

Solution

The volume of the cylinder is $V = \pi r^2 h$, and we are given $\dfrac{dr}{dt} = 3$ and $\dfrac{dh}{dt} = -5$. We find that

$$\frac{\partial V}{\partial r} = \pi(2r)h \quad \text{and} \quad \frac{\partial V}{\partial h} = \pi r^2(1)$$

By the chain rule for one parameter,

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$$

Thus, at the instant when $r = 10$ and $h = 8$, we have

$$\frac{dV}{dt} = 2\pi(10)(8)(3) + \pi(10)^2(-5) = -20\pi$$

The volume is decreasing at the rate of about 62.8 in.^{3}/min. ∎

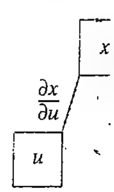
If $F(x, y) = 0$ defines y implicitly as a differentiable function x, we can regard x as a parameter and apply the chain rule to obtain

$$0 = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

so

$$\frac{dy}{dx} = \frac{-\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \qquad \text{provided } F_y \neq 0$$

This formula provides a useful procedure for implicit differentiation. This alternative procedure is illustrated in the following example.



EXAMPLE 4 Implicit differentiation using partial derivatives

If y is a differentiable function of x such that

$$\sin(x+y)+\cos(x-y)=y$$

find $\dfrac{dy}{dx}$.

Solution

Let $F(x, y)=\sin(x+y)+\cos(x-y)-y$, so that $F(x, y)=0$. Then

$$F_x=\cos(x+y)-\sin(x-y)$$
$$F_y=\cos(x+y)-\sin(x-y)(-1)-1$$

so

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=\frac{-[\cos(x+y)-\sin(x-y)]}{\cos(x+y)+\sin(x-y)-1} \quad \blacksquare$$

When z is defined implicitly in terms of x and y by an equation $F(x, y, z)=0$, the chain rule can be used to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in terms of F_x, F_y, and F_z. The procedure is outlined in Problem 57.

EXAMPLE 5 Second derivative of a function of two variables

Let $z=f(x, y)$, where $x=at$ and $y=bt$ for constants a and b. Assuming all necessary differentiability, find d^2z/dt^2 in terms of the partial derivatives of z.

Solution

By using the chain rule, we have

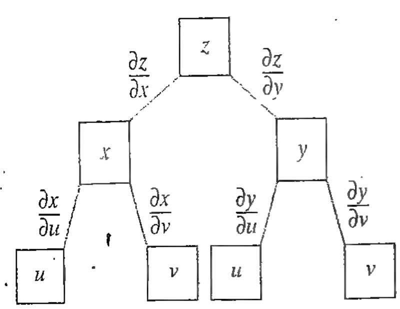
$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

and

$$\frac{d^2z}{dt^2}=\frac{d}{dt}\left(\frac{dz}{dt}\right)=\frac{d}{dt}\left(\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}\right)$$
$$=\frac{\partial z}{\partial x}\left[\frac{d}{dt}\left(\frac{dx}{dt}\right)\right]+\left[\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)\right]\frac{dx}{dt}+\frac{\partial z}{\partial y}\left[\frac{d}{dt}\left(\frac{dy}{dt}\right)\right]+\left[\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)\right]\frac{dy}{dt}$$
$$=\left[\frac{\partial z}{\partial x}\frac{d^2x}{dt^2}+\frac{dx}{dt}\left(\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt}+\frac{\partial^2 z}{\partial x\partial y}\frac{dy}{dt}\right)\right]+\left[\frac{\partial z}{\partial y}\frac{d^2y}{dt^2}+\frac{dy}{dt}\left(\frac{\partial^2 z}{\partial y\partial x}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}\right)\right]$$

Substituting $\dfrac{dx}{dt}=a$ and $\dfrac{dy}{dt}=b$, we obtain

$$\frac{d^2z}{dt^2}=\left[\frac{\partial z}{\partial x}(0)+a\left(\frac{\partial^2 z}{\partial x^2}a+\frac{\partial^2 z}{\partial x\partial y}b\right)\right]+\left[\frac{\partial z}{\partial y}(0)+b\left(\frac{\partial^2 z}{\partial y\partial x}a+\frac{\partial^2 z}{\partial y^2}b\right)\right]$$
$$=a^2\frac{\partial^2 z}{\partial x^2}+2ab\frac{\partial^2 z}{\partial x\partial y}+b^2\frac{\partial^2 z}{\partial y^2} \qquad \text{Note } \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}. \quad \blacksquare$$



EXTENSIONS OF THE CHAIN RULE

Next, we will consider the kind of composite function that occurs when x and y are both functions of two parameters. Specifically, let $z=F(x, y)$, where $x=x(u, v)$ and $y=y(u, v)$ are both functions of two independent parameters u and v. Then $z=F[x(u, v), y(u, v)]$ is a composite function of u and v, and with suitable assumptions

regarding differentiability, we can find the partial derivatives $\partial z/\partial u$ and $\partial z/\partial v$ by applying the chain rule obtained in the following theorem.

THEOREM 11.5 The chain rule for two independent parameters

Suppose $z = f(x, y)$ is differentiable at (x, y) and that the partial derivatives of $x = x(u, v)$ and $y = y(u, v)$ exist at (u, v). Then, the composite function $z = f[x(u, v), y(u, v)]$ is differentiable at (u, v) with

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Proof This version of the chain rule follows immediately from the chain rule for one independent parameter. For example, if v is fixed, the composite function $z = f[x(u, v), y(u, v)]$ depends on u alone, and we have the situation described in the chain rule of one independent variable. We apply this chain rule with a partial derivative (because x and y are functions of more than one variable):

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

The formula for $\dfrac{\partial z}{\partial v}$ can be established in a similar fashion. ❑

EXAMPLE 6 Chain rule for two independent parameters

Let $z = 4x - y^2$, where $x = uv^2$ and $y = u^3v$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Solution

First find the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(4x - y^2) = 4 \qquad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(4x - y^2) = -2y$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(uv^2) = v^2 \qquad \frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(u^3v) = 3u^2v$$

and

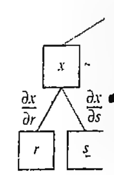
$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(uv^2) = 2uv \qquad \frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(u^3v) = u^3$$

Therefore, the chain rule for two independent parameters gives

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ &= (4)(v^2) + (-2y)(3u^2v) = 4v^2 - 2(u^3v)(3u^2v) = 4v^2 - 6u^5v^2\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\\ &= (4)(2uv) + (-2y)(u^3) = 8uv - 2(u^3v)u^3 = 8uv - 2u^6v\end{aligned}$$ ■



EXAMPLE 7 Implicit differentiation using the chain rule

If f is differentiable and $z = u + f(u^2 v^2)$, show that $u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v} = u$.

Solution

Let $w = u^2v^2$, so $z = u + f(w)$. Then, according to the chain rule,

$$\frac{\partial z}{\partial u} = 1 + \frac{df}{dw}\frac{\partial w}{\partial u} \qquad\qquad \frac{\partial z}{\partial v} = \frac{df}{dw}\frac{\partial w}{\partial v}$$
$$= 1 + f'(w)(2uv^2) \qquad\qquad = f'(w)(2u^2v)$$

so that

$$\begin{aligned} u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} &= u\left[1 + f'(w)(2uv^2)\right] - v\left[f'(w)(2u^2v)\right] \\ &= u + f'(w)\left[u(2uv^2) - v(2u^2v)\right] \\ &= u \end{aligned}$$

■

The chain rules can be extended to functions of three or more variables. For instance, if $w = f(x, y, z)$ is a differentiable function of three variables and $x = x(t)$, $y = y(t)$, $z = z(t)$ are each differentiable functions of t, then w is a differentiable composite function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

In general, if $w = f(x_1, x_2, \ldots, x_n)$ is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$, which in turn are differentiable functions of m parameters $t_1, t_2, \ldots, t_m$, then

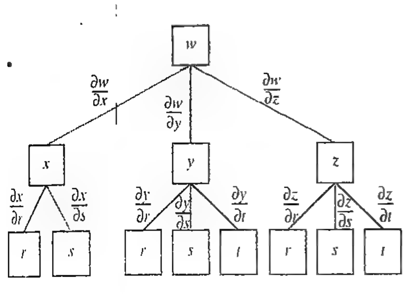
$$\begin{aligned} \frac{\partial w}{\partial t_1} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1} \\ &\vdots \\ \frac{\partial w}{\partial t_m} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m} \end{aligned}$$

EXAMPLE 8 Chain rule for a function of three variables with three parameters

Find $\dfrac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$, where $x = e^{rs^2}$, $y = \ln\dfrac{r+s}{t}$, and $z = rst^2$.

Solution

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= \left[\frac{\partial}{\partial x}(4x + y^2 + z^3)\right]\left[\frac{\partial}{\partial s}(e^{rs^2})\right] + \left[\frac{\partial}{\partial y}(4x + y^2 + z^3)\right]\left[\frac{\partial}{\partial s}\left(\ln\frac{r+s}{t}\right)\right] \\ &\quad + \left[\frac{\partial}{\partial z}(4x + y^2 + z^3)\right]\left[\frac{\partial}{\partial s}(rst^2)\right] \\ &= 4\left[e^{rs^2}(2rs)\right] + 2y\left(\frac{1}{\frac{r+s}{t}}\right)\left(\frac{1}{t}\right) + 3z^2(rt^2) \\ &= 8rse^{rs^2} + \frac{2y}{r+s} + 3rt^2z^2 \end{aligned}$$



In terms of r, s, and t, the partial derivative is

$$\frac{\partial w}{\partial s} = 8rse^{rs^2} + \frac{2}{r+s}\ln\frac{r+s}{t} + 3r^3s^2t^6$$

HW(3)

Guideline § 5-35, 44-49, 54, 52

11.5 PROBLEM SET

A 1. **WHAT DOES THIS SAY?** Discuss the various chain rules and the need for such chain rules.

2. **WHAT DOES THIS SAY?** Discuss the usefulness of the schematic (tree) representation for the chain rules.

3. **WHAT DOES THIS SAY?** Write out a chain rule for a function of two variables and three independent parameters.

In Problems 4–7, the function $z = f(x, y)$ depends on x and y, which in turn are each functions of t. In each case, find dz/dt in two different ways:

a. Express z explicitly in terms of t.

b. Use the chain rule for one parameter.

4. $f(x, y) = 2xy + y^2$, where $x = -3t^2$ and $y = 1 + t^3$
5. $f(x, y) = (4 + y^2)x$, where $x = e^{2t}$ and $y = e^{3t}$
6. $f(x, y) = (1 + x^2 + y^2)^{1/2}$, where $x = \cos 5t$ and $y = \sin 5t$
7. $f(x, y) = xy^2$, where $x = \cos 3t$ and $y = \tan 3t$

In Problems 8–11, the function $F(x, y)$ depends on x and y. Let $x = x(u, v)$ and $y = y(u, v)$ be given functions of u and v. Let $z = F[x(u, v), y(u, v)]$ and find the partial derivatives $\partial z/\partial u$ and $\partial z/\partial v$ in these two ways:

a. Express z explicitly in terms of u and v.

b. Apply the chain rule for two independent parameters.

8. $F(x, y) = x + y^2$, where $x = u + v$ and $y = u - v$
9. $F(x, y) = x^2 + y^2$, where $x = u\sin v$ and $y = u - 2v$
10. $F(x, y) = e^{xy}$, where $x = u - v$ and $y = u + v$
11. $F(x, y) = \ln xy$, where $x = e^{uv^2}$, $y = e^{uv}$

Write out the chain rule for the functions given in Problems 12–15.

12. $z = f(x, y)$, where $x = x(s, t)$, $y = y(s, t)$
13. $w = f(x, y, z)$, where $x = x(s, t)$, $y = y(s, t)$, $z = z(s, t)$
14. $t = f(u, v)$, where $u = u(x, y, z, w)$, $v = v(x, y, z, w)$
15. $w = f(x, y, z)$, where $x = x(s, t, u)$, $y = y(s, t, u)$, $z = z(s, t, u)$

Find the indicated derivatives or partial derivatives in Problems 16–21. Leave your answers in mixed form (x, y, z, t).

16. Find $\dfrac{dw}{dt}$, where $w = \ln(x + 2y - z^2)$ and $x = 2t - 1$, $y = \dfrac{1}{t}$, $z = \sqrt{t}$.

17. Find $\dfrac{dw}{dt}$, where $w = \sin xyz$ and $x = 1 - 3t$, $y = e^{1-t}$, $z = 4t$.

18. Find $\dfrac{dw}{dt}$, where $w = ze^{xy^2}$ and $x = \sin t$, $y = \cos t$, $z = \tan 2t$.

19. Find $\dfrac{dw}{dt}$, where $w = e^{x^3+yz}$ and $x = \dfrac{2}{t}$, $y = \ln(2t - 3)$, $z = t^2$.

20. Find $\dfrac{\partial w}{\partial r}$, where $w = e^{2x-y+3z^2}$ and $x = r + s - t$, $y = 2r - 3s$, $z = \cos rst$.

21. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial t}$, where $w = \dfrac{x+y}{2-z}$ and $x = 2rs$, $y = \sin rt$, $z = st^2$.

In Problems 22–27, assume the given equations define y as a differentiable function of x and find dy/dx using the procedure illustrated in Example 4.

22. $x^2y + \sqrt{xy} = 4$
23. $(x^2 - y)^{3/2} + x^2y = 2$
24. $x^2y + \ln(2x + y) = 5$
25. $x\cos y + y\tan^{-1}x = x$
26. $xe^{xy} + ye^{-xy} = 3$
27. $\tan^{-1}\left(\dfrac{x}{y}\right) = \tan^{-1}\left(\dfrac{y}{x}\right)$

B *Find the following higher-order partial derivatives in Problems 28–33.*

a. $\dfrac{\partial^2 z}{\partial x \partial y}$ **b.** $\dfrac{\partial^2 z}{\partial x^2}$ **c.** $\dfrac{\partial^2 z}{\partial y^2}$

28. $x^3 + y^2 + z^2 = 5$
29. $xyz = 2$
30. $\ln(x + y) = y^2 + z$
31. $x^{-1} + y^{-1} + z^{-1} = 3$
32. $x\cos y = y + z$
33. $z^2 + \sin x = \tan y$

34. Let $f(x, y)$ be a differentiable function of x and y, and let $x = r\cos\theta$, $y = r\sin\theta$ for $r > 0$ and $0 < \theta < 2\pi$.

a. If $z = f[x(r, \theta), y(r, \theta)]$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$.

b. Show that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

35. Let $z = f(x, y)$, where $x = au$ and $y = bv$, with a, b constants. Express $\partial^2 z/\partial u^2$ and $\partial^2 z/\partial v^2$ in terms of the partial derivatives of z with respect to x and y. Assume the existence and continuity of all necessary first and second partial derivatives.

36. Let (x, y, z) lie on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Without solving for z explicitly in terms of x and y, compute the higher-order partial derivatives

$$\frac{\partial^2 z}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y}$$

37. The dimensions of a rectangular box are linear functions of time, $\ell(t)$, $w(t)$, and $h(t)$. If the length and width are increasing at 2 in./sec and the height is decreasing at 3 in./sec, find the rates at which the volume V and the surface area S are changing with respect to time. If $\ell(0) = 10$, $w(0) = 8$, and $h(0) = 20$, is V increasing or decreasing when $t = 5$ sec? What about S when $t = 5$?

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38. Van der Waal's equation in physical chemistry states that a gas occupying volume V at temperature T (Kelvin) exerts pressure P, where

$$\left(P + \frac{A}{V^2}\right)(V - B) = kT$$

for physical constants A, B, and k. Compute the following rates:

a. the rate of change of volume with respect to temperature

b. the rate of change of pressure with respect to volume

39. The concentration of a drug in the blood of a patient t hours after the drug is injected into the body intramuscularly is modeled by the Heinz function

$$C = \frac{1}{b-a}(e^{-at} - e^{-bt}) \qquad b > a$$

where a and b are parameters that depend on the patient's metabolism and the particular kind of drug being used.

a. Compute the rates $\dfrac{\partial C}{\partial a}$, $\dfrac{\partial C}{\partial b}$, and $\dfrac{\partial C}{\partial t}$.

b. Explore the assumption that $a = (\ln b)/t$, b constant for $t > (\ln b)/b$. In particular, what is dC/dt?

40. A paint store carries two brands of latex paint. An analysis of sales figures indicates that the demand Q for the first brand is modeled by

$$Q(x, y) = 210 - 12x^2 + 18y$$

gallons/month, where x, y are the prices of the first and second brands, respectively. A separate study indicates that t months from now, the first brand will cost $x = 4 + 0.18t$ dollars/gal and the second brand will cost $y = 5 + 0.3\sqrt{t}$ dollars/gal. At what rate will the demand Q be changing with respect to time 9 months from now?

41. To model the demand for the sale of bicycles, it is assumed that if 24-speed bicycles are sold for x dollars apiece and the price of gasoline is y cents per gallon, then

$$Q(x, y) = 240 - 21\sqrt{x} + 4(0.2y + 12)^{3/2}$$

bicycles will be sold each month. For this model, it is further assumed that t months from now, bicycles will be selling for $x = 120 + 6t$ dollars apiece, and the price of gasoline will be $y = 80 + 10\sqrt{4t}$ cents/gal. At what rate will the monthly demand for the bicycles be changing with respect to time 4 months from now?

42. At a certain factory, the amount of air pollution generated each day is modeled by the function

$$Q(E, T) = 127E^{2/3}T^{1/2}$$

where E is the number of employees and T (°C) is the average temperature during the workday. Currently, there are 142 employees and the average temperature is 18°C. If the average daily temperature is falling at the rate of 0.23°/day and the number of employees is increasing at the rate of 3/month, what is the corresponding effect on the rate of pollution? Express your answer in units/day. For this model, assume there are 22 workdays/month.

43. The combined resistance R produced by three variable resistances R_1, R_2, and R_3 connected in parallel is modeled by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Suppose at a certain instant, $R_1 = 100$ ohms, $R_2 = 200$ ohms, $R_3 = 300$ ohms, and that R_1 and R_3 are decreasing at the rate of 1.5 ohm/s while R_2 is increasing at the rate of 2 ohms/s. How fast is R changing with respect to time at this instant? Is it increasing or decreasing?

In Problems 44–48, assume that all functions have whatever derivatives or partial derivatives are necessary for the problem to be meaningful.

44. If $z = f(uv^2)$, show that

$$2u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = 0$$

45. If $z = f(u - v, v - u)$, show that

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = 0$$

46. If $z = u + f(uv)$, show that

$$u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = u$$

47. If $w = f\left(\dfrac{r-s}{s}\right)$, show that

$$r\frac{\partial w}{\partial r} + s\frac{\partial w}{\partial s} = 0$$

48. If $z = xy + f(x^2 + y^2)$, show that

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y^2 - x^2$$

49. Let $w = f(t)$ be a differentiable function of t, where $t = (x^2 + y^2 + z^2)^{1/2}$. Show that

$$\left(\frac{dw}{dt}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2$$

50. Suppose f is a twice differentiable function of one variable, and let $z = f(x^2 + y^2)$. Find

a. $\dfrac{\partial^2 z}{\partial x^2}$ b. $\dfrac{\partial^2 z}{\partial y^2}$ c. $\dfrac{\partial^2 z}{\partial x \partial y}$

51. Find $\dfrac{d^2 z}{d\theta^2}$, where z is a twice differentiable function of one variable θ and can be written $z = f(\cos\theta, \sin\theta)$. *Hint:* Let $x = \cos\theta$ and $y = \sin\theta$. Leave your answer in terms of x and y.

52. Let f and g be twice differentiable functions of one variable, and let

$$u(x, t) = f(x + ct) + g(x - ct)$$

for a constant c. Show that $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$. *Hint:* Let $r = x + ct$; $s = x - ct$.

53. Suppose $z = f(x, y)$ has continuous second-order partial derivatives. If $x = e^r\cos\theta$ and $y = e^r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2r}\left[\frac{\partial^2 z}{\partial r^2} + \frac{\partial^2 z}{\partial \theta^2}\right]$$

54. If $f(u, v, w)$ is differentiable and $u = x - y$, $v = y - z$, and $w = z - x$, what is

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}?$$

55. The *Cauchy-Riemann equations* are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(See Problem 51, Section 11.3.) Show that if x and y are expressed in terms of polar coordinates, the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

56. Let $T(x, y)$ be the temperature at each point (x, y) in a portion of the plane that contains the ellipse $x = 2\cos t$, $y = \sin t$ for $0 \le t \le 2\pi$. Suppose

$$\frac{\partial T}{\partial x} = y \quad \text{and} \quad \frac{\partial T}{\partial y} = x$$

a. Find $\frac{dT}{dt}$ and $\frac{d^2T}{dt^2}$ by using the chain rule.

b. Locate the maximum and minimum temperatures on the ellipse.

57. Let $F(x, y, z)$ be a function of three variables with continuous partial derivatives F_x, F_y, F_z in a certain region where $F(x, y, z) = C$ for some constant C. Use the chain rule for two parameters and the fact that x and y are independent variables to show that (for $F_z \neq 0$)

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Use these formulas to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly by the equation $x^2 + 2xyz + y^3 + e^z = 4$.

58. Suppose the system

$$\begin{cases} xu + yv - uv = 0 \\ yu - xv + uv = 0 \end{cases}$$

can be solved for u and v in terms of x and y, so that $u = u(x, y)$ and $v = v(x, y)$. Use implicit differentiation to find the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$.

59. A function $f(x, y)$ is said to be *homogeneous of degree n* if

$$f(tx, ty) = t^n f(x, y) \qquad \text{for all } t > 0$$

a. Show that $f(x, y) = x^2y + 2y^3$ is homogeneous and find its degree.

b. If $f(x, y)$ is homogeneous of degree n, show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

60. Suppose that F and G are functions of three variables and that it is possible to solve the equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ for y and z in terms of x, so that $y = y(x)$ and $z = z(x)$. Use the chain rule to express dy/dx and dz/dx in terms of the partial derivatives of F and G. Assume these partials are continuous and that

$$\frac{\partial F}{\partial y}\frac{\partial G}{\partial z} \neq \frac{\partial F}{\partial z}\frac{\partial G}{\partial y}$$

11.6 Directional Derivatives and the Gradient

IN THIS SECTION the directional derivative, the gradient, maximal property of the gradient, functions of three variables, normal property of the gradient, tangent planes and normal lines

Suppose $z = T(x, y)$ gives the temperature at each point (x, y) in a region R of the plane, and let $P_0(x_0, y_0)$ be a particular point in R. Then we know that the partial derivative $T_x(x_0, y_0)$ gives the rate at which the temperature changes for a move from P_0 in the x-direction, while the rate of temperature change in the y-direction is given by $T_y(x_0, y_0)$. Suppose we want to find the direction of greatest temperature change, which may be in a direction not parallel to either coordinate axes. To answer this question, we will introduce the concept of *directional derivative* in this section and examine its properties.

Direction

WARNING

unit vector

THE DIRECTIONAL DERIVATIVE

In Chapter 3 we defined the *slope of a curve* at a point to be the ratio of the change in the dependent variable to the change in the independent variable at the given point. To determine the slope of the tangent line at a point $P_0(x_0, y_0)$ on a surface defined by $z = f(x, y)$, we need to specify the *direction* in which we wish to measure. We do this

by using vectors. In Section 11.3 we found the slope parallel to the xz-plane to be the partial derivative $f_x(x_0, y_0)$. We could have specified this direction in terms of the unit vector $\mathbf{i}$ (x-direction), while $f_y(x, y)$ could have been specified in terms of the unit vector $\mathbf{j}$. Finally, to measure the slope of the tangent line in an *arbitrary* direction, we use a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ in that direction.

To find the desired slope, we look at the intersection of the surface with the vertical plane passing through the point P_0 parallel to the vector $\mathbf{u}$, as shown in Figure 11.26. This vertical plane intersects the surface to form a curve C, and we define the slope of the surface at P_0 in the direction of $\mathbf{u}$ to be the slope of the tangent line to the curve C defined by $\mathbf{u}$ at that point.

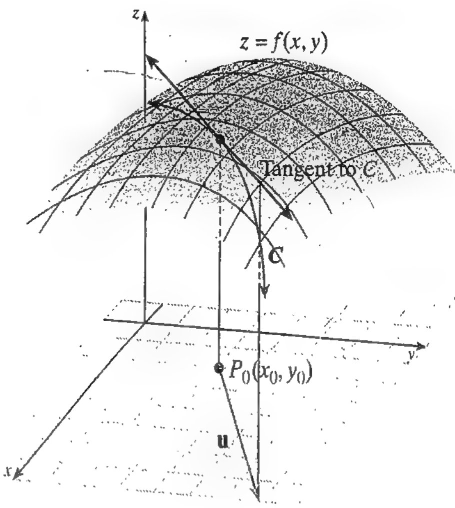


Figure 11.26 The directional derivative $D_u f(x_0, y_0)$ is the slope of the tangent line to the curve on the surface $z = f(x, y)$ in the direction of the unit vector $\mathbf{u}$ at $P_0(x_0, y_0)$.

We summarize this idea of slope *in a particular direction* with the following definition.

Directional Derivative

WARNING Remember, $\mathbf{u}$ must be a *unit* vector.

Let f be a function of two variables, and let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ be a unit vector. The **directional derivative of f at $P_0(x_0, y_0)$** in the direction of $\mathbf{u}$ is given by

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

provided the limit exists.

At a particular point $P_0(x_0, y_0)$, there are infinitely many directional derivatives to the graph of $z = f(x, y)$, one for each direction radiating from P_0. Two of these are the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$. To see this, note that if $\mathbf{u} = \mathbf{i}$ (so $u_1 = 1$ and $u_2 = 0$), then

$$D_{\mathbf{i}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = f_x(x_0, y_0)$$

and if $\mathbf{u} = \mathbf{j}$ (so $u_1 = 0$ and $u_2 = 1$),

$$D_{\mathbf{j}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} = f_y(x_0, y_0)$$

The definition of the directional derivative is similar to the definition of the derivative of a function of a single variable. Just as with a single variable, it is difficult to apply the definition directly. Fortunately, the following theorem allows us to find directional derivatives more efficiently than by using the definition.

THEOREM 11.6 Directional derivatives using partial derivatives

Gradient

Let $f(x, y)$ be a function that is differentiable at $P_0(x_0, y_0)$. Then f has a directional derivative in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ given by

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

Proof We define a function F of a single variable h by $F(h) = f(x_0 + hu_1, y_0 + hu_2)$, so that

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$
$$= \lim_{h\to 0} \frac{F(h) - F(0)}{h} = F'(0)$$

Apply the chain rule with $x = x_0 + hu_1$ and $y = y_0 + hu_2$:

$$F'(h) = \frac{dF}{dh} = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)u_1 + f_y(x, y)u_2$$

When $h = 0$, we have $x = x_0$ and $y = y_0$, so that

$$D_{\mathbf{u}}f(x_0, y_0) = F'(0) = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \quad \square$$

EXAMPLE 1 Finding a directional derivative using partial derivatives

Find the directional derivative of $f(x, y) = 3 - 2x^2 + y^3$ at the point $P(1, 2)$ in the direction of the unit vector $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$.

Solution

First, find the partial derivatives $f_x(x, y) = -4x$ and $f_y(x, y) = 3y^2$. Then, since $u_1 = \frac{1}{2}$ and $u_2 = -\frac{\sqrt{3}}{2}$, we have

$$D_{\mathbf{u}}f(1, 2) = f_x(1, 2)\left(\frac{1}{2}\right) + f_y(1, 2)\left(-\frac{\sqrt{3}}{2}\right)$$
$$= -4(1)\left(\frac{1}{2}\right) + 3(2)^2\left(-\frac{\sqrt{3}}{2}\right) = -2 - 6\sqrt{3} \approx -12.4 \quad \blacksquare$$

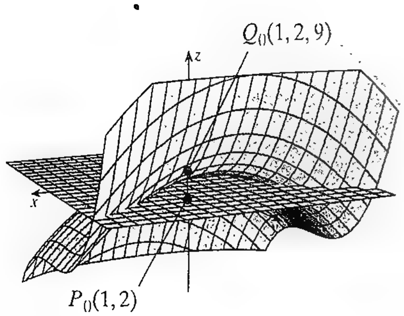


Figure 11.27 The graph of the surface $z = 3 - 2x^2 + y^3$

The directional derivative of $f(x, y)$ at the point $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2\rangle$ can be interpreted as both a rate of change and a slope. For instance, in Example 1, the intersection of the surface $z = 3 - 2x^2 + y^3$ with the vertical plane through the point $P(1, 2)$ parallel to the unit vector $\mathbf{u} = \left\langle \frac{1}{2}, \frac{-\sqrt{3}}{2} \right\rangle$ is a curve C, and the directional derivative $D_{\mathbf{u}}f(1, 2) = -12.3$ is the slope of C at the point $Q(1, 2, 9)$ on the surface above P, as shown in Figure 11.27. The directional derivative also gives the rate at which the function $f(x, y) = 3 - 2x^2 + y^3$ changes as a point (x, y) moves from P in the direction of $\mathbf{u}$.

THE GRADIENT

The directional derivative $D_{\mathbf{u}}f(x, y)$ can be expressed concisely in terms of a vector function called the *gradient*, which has many important uses in mathematics. The gradient of a function of two variables may be defined as follows.

Gradient

> Let f be a differentiable function at (x, y) and let $f(x, y)$ have partial derivatives $f_x(x, y)$ and $f_y(x, y)$. Then the **gradient** of f, denoted by ∇f (pronounced "del eff"), is a vector given by
>
> $$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$
>
> The value of the gradient at the point $P_0(x_0, y_0)$ is denoted by
>
> $$\nabla f_0 = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$$

Note: Think of the symbol ∇ as an "operator" on a function that produces a vector. Another notation for ∇f is **grad** $f(x, y)$.

EXAMPLE 2 Finding the gradient of a given function

Find $\nabla f(x, y)$ for $f(x, y) = x^2y + y^3$.

Solution

Begin with the partial derivatives:

$$f_x(x, y) = \frac{\partial}{\partial x}(x^2y + y^3) = 2xy \quad \text{and} \quad f_y(x, y) = \frac{\partial}{\partial y}(x^2y + y^3) = x^2 + 3y^2$$

Then

$$\nabla f(x, y) = 2xy\mathbf{i} + (x^2 + 3y^2)\mathbf{j}$$ ■

The following theorem shows how the directional derivative can be expressed in terms of the gradient.

THEOREM 11.7 **The gradient formula for the directional derivative**

If f is a differentiable function of x and y, then the directional derivative of f at the point $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f_0 \cdot \mathbf{u}$$

Proof Because $\nabla f_0 = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$ and $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, we have

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f_0 \cdot \mathbf{u} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$ ❑

EXAMPLE 3 Using the gradient formula to compute a directional derivative

Find the directional derivative of $f(x, y) = \ln(x^2 + y^3)$ at $P_0(1, -3)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$.

Solution

$$f_x(x, y) = \frac{2x}{x^2 + y^3}, \quad \text{so} \quad f_x(1, -3) = -\frac{2}{26}$$

$$f_y(x, y) = \frac{3y^2}{x^2 + y^3}, \quad \text{so} \quad f_y(1, -3) = -\frac{27}{26}$$

$$\nabla f_0 = \nabla f(1, -3) = -\tfrac{2}{26}\mathbf{i} - \tfrac{27}{26}\mathbf{j}$$

A unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2\mathbf{i} - 3\mathbf{j}}{\sqrt{2^2 + (-3)^2}} = \frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{j})$$

Thus,

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= \nabla f \cdot \mathbf{u} = \left(-\frac{2}{26}\right)\left(\frac{2}{\sqrt{13}}\right) + \left(-\frac{27}{26}\right)\left(-\frac{3}{\sqrt{13}}\right) \\ &= \frac{77\sqrt{13}}{338} \end{aligned}$$

Although a differentiable function of one variable $f(x)$ has exactly one derivative $f'(x)$, a differentiable function of two variables $F(x, y)$ has two partial derivatives and an infinite number of directional derivatives. Is there any single mathematical concept for functions of several variables that is the analogue of the derivative of a function of a single variable? The properties listed in the following theorem suggest that the gradient plays this role.

THEOREM 11.8 Basic properties of the gradient

Let f and g be differentiable functions. Then

Constant rule	$\nabla c = \mathbf{0}$ for any constant c
Linearity rule	$\nabla(af + bg) = a\nabla f + b\nabla g$ for constants a and b
Product rule	$\nabla(fg) = f\nabla g + g\nabla f$
Quotient rule	$\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$, $g \neq 0$
Power rule	$\nabla(f^n) = nf^{n-1}\nabla f$

Proof **Linearity rule**

$$\begin{aligned} \nabla(af + bg) &= (af + bg)_x\mathbf{i} + (af + bg)_y\mathbf{j} = (af_x + bg_x)\mathbf{i} + (af_y + bg_y)\mathbf{j} \\ &= af_x\mathbf{i} + bg_x\mathbf{i} + af_y\mathbf{j} + bg_y\mathbf{j} = a(f_x\mathbf{i} + f_y\mathbf{j}) + b(g_x\mathbf{i} + g_y\mathbf{j}) \\ &= a\nabla f + b\nabla g \end{aligned}$$

Power rule

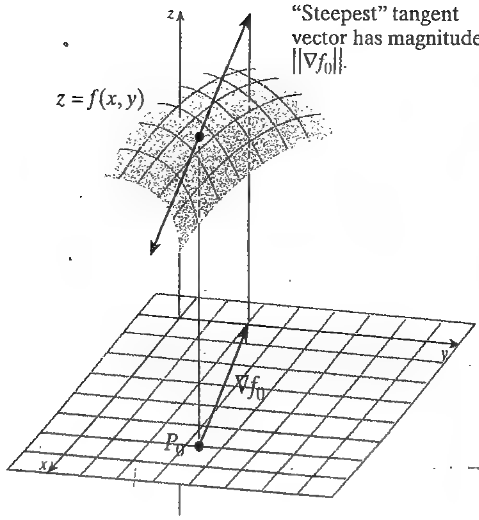
$$\begin{aligned} \nabla f^n &= [f^n]_x\mathbf{i} + [f^n]_y\mathbf{j} = nf^{n-1}f_x\mathbf{i} + nf^{n-1}f_y\mathbf{j} \\ &= nf^{n-1}[f_x\mathbf{i} + f_y\mathbf{j}] = nf^{n-1}\nabla f \end{aligned}$$

The other rules are left for the problem set (Problem 57). ❑

MAXIMAL PROPERTY OF THE GRADIENT

In applications, it is often useful to compute the greatest rate of increase (or decrease) of a given function at a specified point. The direction in which this occurs is called the direction of **steepest ascent** (or **steepest descent**). For example, suppose the function $z = f(x, y)$ gives the altitude of a skier coming down a slope, and we want to state a theorem that will give the skier the *compass direction* of the path of steepest descent (see Figure 11.28b). We emphasize the words "compass direction" because the gradient gives direction in the xy-plane and does not itself point up or down the mountain. The following theorem shows how the direction of maximum change is determined by the gradient (see Figure 11.28).

Figure 11
descent



a. The optimal direction property of the gradient

b. Skier on a slope

Figure 11.28 Steepest ascent or steepest descent

***THEOREM 11.9* Maximal direction property of the gradient**

Suppose f is differentiable at the point P_0 and that the gradient of f at P_0 satisfies $\nabla f_0 \neq \mathbf{0}$. Then

a. The largest value of the directional derivative $D_{\mathbf{u}} f$ at P_0 is $\|\nabla f_0\|$ and occurs when the unit vector $\mathbf{u}$ points in the direction of ∇f_0.

b. The smallest value of $D_{\mathbf{u}} f$ at P_0 is $-\|\nabla f_0\|$ and occurs when $\mathbf{u}$ points in the direction of $-\nabla f_0$.

Proof If $\mathbf{u}$ is any unit vector, then

$$D_{\mathbf{u}} f = \nabla f_0 \cdot \mathbf{u} = \|\nabla f_0\| \, (\|\mathbf{u}\| \cos\theta) = \|\nabla f_0\| \cos\theta$$

where θ is the angle between ∇f_0 and $\mathbf{u}$. But $\cos\theta$ assumes its largest value 1 at $\theta = 0$; that is, when $\mathbf{u}$ points in the direction ∇f_0. Thus, the largest possible value of $D_{\mathbf{u}} f$ is

$$D_{\mathbf{u}} f = \|\nabla f_0\|(1) = \|\nabla f_0\|$$

Statement **b** may be established in a similar fashion by noting that $\cos\theta$ assumes its smallest value -1 when $\theta = \pi$. This value occurs when $\mathbf{u}$ points toward $-\nabla f_0$, and in this direction

$$D_{\mathbf{u}} f = \|\nabla f_0\|(-1) = -\|\nabla f_0\|$$

□

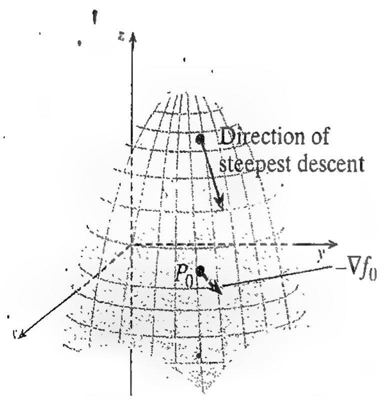


Figure 11.29 The direction of steepest descent

➡ **What This Says** The theorem states that at P_0 the function f increases most rapidly in the direction of the gradient ∇f_0 and decreases most rapidly in the opposite direction (see Figure 11.29).

EXAMPLE 4 Maximal rate of increase and decrease

In what direction is the function defined by $f(x, y) = xe^{2y-x}$ increasing most rapidly at the point $P_0(2, 1)$, and what is the maximum rate of increase? In what direction is f decreasing most rapidly?

Solution

We begin by finding the gradient of f:

$$\begin{aligned}\nabla f = f_x\mathbf{i} + f_y\mathbf{j} &= [e^{2y-x} + xe^{2y-x}(-1)]\mathbf{i} + [xe^{2y-x}(2)]\mathbf{j}\\ &= e^{2y-x}[(1-x)\mathbf{i} + 2x\mathbf{j}]\end{aligned}$$

At $(2, 1)$, $\nabla f_0 = e^{2(1)-2}\,[(1-2)\mathbf{i} + 2(2)\mathbf{j}] = -\mathbf{i} + 4\mathbf{j}$. The most rapid rate of increase is $\|\nabla f_0\| = \sqrt{(-1)^2 + (4)^2} = \sqrt{17}$ and it occurs in the direction of $-\mathbf{i} + 4\mathbf{j}$. The most rapid rate of decrease occurs in the direction of $-\nabla f_0 = \mathbf{i} - 4\mathbf{j}$. ■

FUNCTIONS OF THREE VARIABLES

The directional derivative and gradient concepts can easily be extended to functions of three or more variables. For a function of three variables, $f(x, y, z)$, the gradient ∇f is defined by

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$$

and the directional derivative $D_{\mathbf{u}}f$ of $f(x, y, z)$ at $P_0(x_0, y_0, z_0)$ in the direction of the unit vector $\mathbf{u}$ is given by

$$D_{\mathbf{u}}f = \nabla f_0 \cdot \mathbf{u}$$

where, as before, ∇f_0 is the gradient ∇f evaluated at P_0. The basic properties of the gradient of $f(x, y)$ (Theorem 11.8) are still valid, as is the maximal direction property of Theorem 11.9. Similar definitions and properties are valid for functions of more than three variables.

EXAMPLE 5 Directional derivative of a function of three variables

Let $f(x, y, z) = xy\sin(xz)$. Find ∇f_0 at the point $P_0(1, -2, \pi)$ and then compute the directional derivative of f at P_0 in the direction of the vector $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$.

Solution

Begin with the partial derivatives:

$$\begin{aligned}
f_x &= y\sin(xz) + xy(z\cos(xz)); & f_x(1, -2, \pi) &= -2\sin\pi - 2\pi\cos\pi = 2\pi\\
f_y &= x\sin(xz); & f_y(1, -2, \pi) &= 1\sin\pi = 0\\
f_z &= xy(x\cos(xz)); & f_z(1, -2, \pi) &= (1)(-2)(1)\cos\pi = 2
\end{aligned}$$

Thus, the gradient of f at P_0 is

$$\nabla f_0 = 2\pi\mathbf{i} + 2\mathbf{k}$$

To find $D_{\mathbf{u}}f$ we need $\mathbf{u}$, the unit vector in the direction of $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{-2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}}{\sqrt{(-2)^2 + (3)^2 + (-5)^2}} = \frac{1}{\sqrt{38}}(-2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$$

Finally,

$$D_{\mathbf{u}}f(1, -2, \pi) = \nabla f_0 \cdot \mathbf{u} = \frac{1}{\sqrt{38}}(-4\pi - 10) \approx -3.66$$ ■

NORMAL PROPERTY OF THE GRADIENT

Suppose S is a level surface of the function defined by $f(x, y, z)$; that is, $f(x, y, z) = k$ for some constant K. Then if $P_0(x_0, y_0, z_0)$ is a point on S, the following theorem shows that the gradient ∇f_0 at P_0 is a vector that is **normal** (that is, orthogonal) to the tangent plane surface at P_0 (see Figure 11.30).

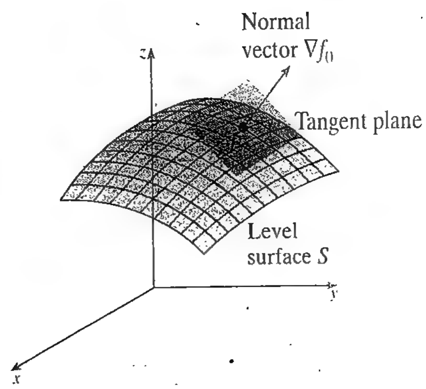


Figure 11.30 The normal property of the gradient

***THEOREM 11.10* The normal property of the gradient**

Suppose the function f is differentiable at the point P_0 and that the gradient at P_0 satisfies $\nabla f_0 \neq \mathbf{0}$. Then ∇f_0 is orthogonal to the level surface of f through P_0.

Proof Let C be any smooth curve on the level surface $f(x, y, z) = K$ that passes through $P_0(x_0, y_0, z_0)$, and describe the curve C by the vector function $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ for all t in some interval I. We will show that the gradient ∇f_0 is orthogonal to the tangent vector $d\mathbf{R}/dt$ at P_0.

Because C lies on the level surface, any point $P(x(t), y(t), z(t))$ on C must satisfy $f[x(t), y(t), z(t)] = K$, and by applying the chain rule, we obtain

$$\frac{d}{dt}[f(x(t), y(t), z(t))] = f_x(x, y, z)\frac{dx}{dt} + f_y(x, y, z)\frac{dy}{dt} + f_z(x, y, z)\frac{dz}{dt}$$

Suppose $t = t_0$ at P_0. Then

$$\frac{d}{dt}[f(x(t), y(t), z(t))]\Big|_{t=t_0}$$
$$= f_x(x(t_0), y(t_0), z(t_0))\frac{dx}{dt} + f_y(x(t_0), y(t_0), z(t_0))\frac{dy}{dt} + f_z(x(t_0), y(t_0), z(t_0))\frac{dz}{dt}$$
$$= \nabla f_0 \cdot \frac{d\mathbf{R}}{dt}$$

since $\dfrac{d\mathbf{R}}{dt} = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} + \dfrac{dz}{dt}\mathbf{k}$. We also know that $f(x(t), y(t), z(t)) = K$ for all t in I (because the curve C lies on the level surface $f(x, y, z) = K$). Thus, we have

$$\frac{d}{dt}\{f[x(t), y(t), z(t)]\} = \frac{d}{dt}(K) = 0$$

and it follows that $\nabla f_0 \cdot \dfrac{d\mathbf{R}}{dt} = 0$. We are given that $\nabla f_0 \neq \mathbf{0}$, and $d\mathbf{R}/dt \neq \mathbf{0}$ because the curve C is smooth. Therefore, ∇f_0 is orthogonal to $d\mathbf{R}/dt$, as required. ❑

⇨ **What This Says** The gradient ∇f_0 at each point P_0 on the surface $f(x, y, z) = K$ is orthogonal at P_0 to the tangent vector $\mathbf{T} = \dfrac{d\mathbf{R}}{dt}$ of each curve C on the surface that passes through P_0. Thus, all these tangent vectors lie in a single plane through P_0 with normal vector $\mathbf{N} = \nabla f_0$. This plane is the *tangent plane* to the surface at P_0.

EXAMPLE 6 Finding a vector that is normal to a level surface

Find a vector that is normal to the level surface $x^2 + 2xy - yz + 3z^2 = 7$ at the point $P_0(1, 1, -1)$.

Solution

Since the gradient vector at P_0 is perpendicular to the level surface, we have

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} = (2x + 2y)\mathbf{i} + (2x - z)\mathbf{j} + (6z - y)\mathbf{k}$$

At the point $(1, 1, -1)$, $\nabla f_0 = 4\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$ is the required normal. ■

Here is an example in which f involves only two variables, so $f(x, y) = K$ is a level curve in the plane instead of a level surface in space.

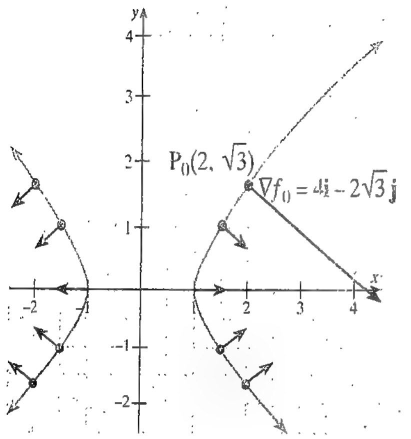


Figure 11.31 The level curve $x^2 - y^2 = 1$

EXAMPLE 7 Finding a vector normal to a level curve

Sketch the level curve corresponding to $C = 1$ for the function $f(x, y) = x^2 - y^2$ and find a normal vector at the point $P_0(2, \sqrt{3})$.

Solution

The level curve for $C = 1$ is a hyperbola given by $x^2 - y^2 = 1$, as shown in Figure 11.31. The gradient vector is perpendicular to the level curve. We have

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} = 2x\mathbf{i} - 2y\mathbf{j}$$

so at the point $(2, \sqrt{3})$, $\nabla f_0 = 4\mathbf{i} - 2\sqrt{3}\mathbf{j}$ is the required normal. This normal vector and a few others are shown in Figure 11.31. ■

EXAMPLE 8 Heat flow application

The set of points (x, y) with $0 \le x \le 5$ and $0 \le y \le 5$ is a square in the first quadrant of the xy-plane. Suppose this square is heated in such a way that $T(x, y) = x^2 + y^2$ is the temperature at the point $P(x, y)$. In what direction will heat flow from the point $P_0(3, 4)$?

Solution

The flow of heat in the region is given by a vector function $\mathbf{H}(x, y)$, whose value at each point (x, y) depends on x and y. From physics it is known that $\mathbf{H}(x, y)$ will be perpendicular to the isothermal curves $T(x, y) = C$ for C constant. The gradient ∇T and all its multiples point in such a direction. Therefore, we can express the heat flow as $\mathbf{H} = -k\nabla T$, where k is a positive constant (called the *thermal conductivity*). The negative sign is introduced to account for the fact that heat flows "downhill" (that is, in the direction of decreasing temperature).

Because $T(3, 4) = 25$, the point $P_0(3, 4)$ lies on the isotherm $T(x, y) = 25$, which is part of the circle $x^2 + y^2 = 25$, as shown in Figure 11.32. We know that the heat flow $\mathbf{H}_0$ at P_0 will satisfy $\mathbf{H}_0 = -k\nabla T_0$, where ∇T_0 is the gradient at P_0. Because $\nabla T = 2x\mathbf{i} + 2y\mathbf{j}$, we see that $\nabla T_0 = 6\mathbf{i} + 8\mathbf{j}$. Thus, the heat flow at P_0 satisfies

$$\mathbf{H}_0 = -k\nabla T_0 = -k(6\mathbf{i} + 8\mathbf{j})$$

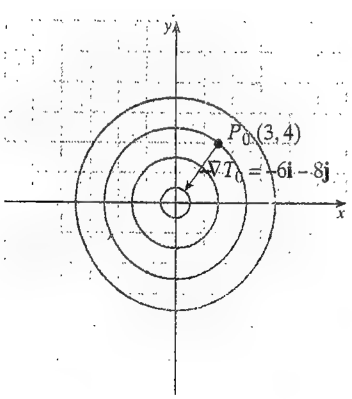


Figure 11.32 Isotherms of $T(x, y) = x^2 + y^2$. Heat flow at P_0 is in the direction of $-\nabla T_0 = -6\mathbf{i} - 8\mathbf{j}$

Because the thermal conductivity k is positive, we can say that heat flows from P_0 in the direction of the unit vector $\mathbf{u}$ given by

$$\mathbf{u} = \frac{-(6\mathbf{i} + 8\mathbf{j})}{\sqrt{(-6)^2 + (-8)^2}} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

■

TANGENT PLANES AND NORMAL LINES

Tangent planes and normal lines to a surface are the natural extensions to $\mathbb{R}^3$ of the tangent and normal lines we examined in $\mathbb{R}^2$. Suppose S is a surface and $\mathbf{N}$ is a vector normal to S at the point P_0. We would intuitively expect the normal line and the tangent plane to S at P_0 to be, respectively, the line through P_0 with the direction of $\mathbf{N}$ and the plane through P_0 with normal $\mathbf{N}$ (see Figure 11.33). These observations lead us to the following definition.

Tangent Plane and Normal Line to a Surface

> Suppose the surface S has a nonzero normal vector $\mathbf{N}$ at the point P_0. Then the line through P_0 parallel to $\mathbf{N}$ is called the **normal line** to S at P_0, and the plane through P_0 with normal vector $\mathbf{N}$ is the **tangent plane** to S at P_0.

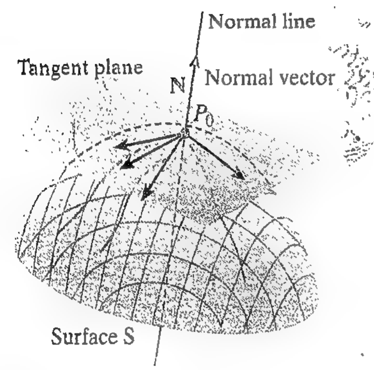


Figure 11.33 Tangent plane and normal line

We would expect a surface S with the representation $z = f(x, y)$ to have a non-vertical tangent plane at each point where $\nabla f \neq \mathbf{0}$. In particular, if S has an equation of the form $F(x, y, z) = C$, where C is a constant and F is a function differentiable at P_0, the normal property of a gradient tells us that the gradient ∇F_0 at P_0 is normal to S (if $\nabla F_0 \neq \mathbf{0}$) and that S must therefore have a tangent plane at P_0.

EXAMPLE 9 Finding the tangent plane and normal line to a given surface

Find equations for the tangent plane and the normal line at the point $P_0(1, -1, 2)$ on the surface S given by $x^2y + y^2z + z^2x = 5$.

Solution

We need to rewrite this problem so that the normal property of the gradient theorem applies. Let $F(x, y, z) = x^2y + y^2z + z^2x$, and consider S to be the level surface $F(x, y; z) = 5$. The gradient ∇F is normal to S at P_0. We find that

$$\nabla F(x, y, z) = (2xy + z^2)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2xz)\mathbf{k}$$

so a normal vector at P_0 is

$$\mathbf{N} = \nabla F_0 = \nabla F(1, -1, 2) = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

Hence, the required tangent plane is

$$2(x - 1) - 3(y + 1) + 5(z - 2) = 0 \quad \text{or} \quad 2x - 3y + 5z = 15$$

The normal line to the surface at P_0 is

$$x = 1 + 2t, \qquad y = -1 - 3t, \qquad z = 2 + 5t$$ ■

By generalizing the procedure illustrated in the preceding example, we are led to the following formulas for the tangent plane and normal line. (Also see Figure 11.34.)

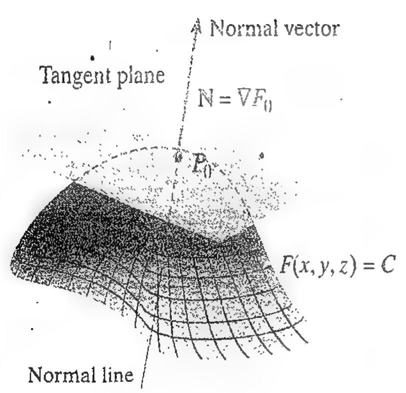


Figure 11.34 The tangent plane and normal line to a surface

> **Formulas for the Tangent Plane and Normal Lines to a Surface**
>
> Suppose S is a surface with the equation $F(x, y, z) = C$ and let $P_0(x_0, y_0, z_0)$ be a point on S where F is differentiable with $\nabla F_0 \neq \mathbf{0}$. Then the **equation of the tangent plane** to S at P_0 is
>
> $$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$
>
> and the **normal line** to S at P_0 has parametric equations
>
> $$x = x_0 + F_x(x_0, y_0, z_0)t$$
> $$y = y_0 + F_y(x_0, y_0, z_0)t$$
> $$z = z_0 + F_z(x_0, y_0, z_0)t$$

Note if $z = f(x, y)$, we have $F(x, y, z) = f(x, y) - z = 0$. Then $F_x = f_x$, $F_y = f_y$, and $F_z = -1$ and the equation of the tangent plane becomes

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) - (z - z_0) = 0$$

which is equivalent to the tangent plane formula given in Section 11.4.

EXAMPLE 10 Equations of the tangent plane and the normal line

Find the equations for the tangent plane and the normal line to the cone $z^2 = x^2 + y^2$ at the point where $x = 3$, $y = 4$, and $z > 0$.

Solution

If $P_0(x_0, y_0, z_0)$ is the point of tangency and $x_0 = 3$, $y_0 = 4$, and $z_0 > 0$, then

$$z_0 = \sqrt{x_0^2 + y_0^2} = \sqrt{9 + 16} = 5$$

If we consider $F(x, y, z) = x^2 + y^2 - z^2$, then the cone can be regarded as the level surface $F(x, y, z) = 0$. The partial derivatives of F are

$$F_x = 2x \qquad F_y = 2y \qquad F_z = -2z$$

so at $P_0(3, 4, 5)$,

$$F_x(3, 4, 5) = 6, \qquad F_y(3, 4, 5) = 8, \qquad F_z(3, 4, 5) = -10$$

Thus the tangent plane has the equation

$$6(x - 3) + 8(y - 4) - 10(z - 5) = 0$$

or $3x + 4y - 5z = 0$, and the normal line is given parametrically by the equations

$$x = 3 + 6t, \qquad y = 4 + 8t, \qquad z = 5 - 10t$$ ■

11.6 PROBLEM SET

Guidelines
8.1–43, 44, 50

A *Find the gradient of the functions given in Problems 1–10.*

1. $f(x, y) = x^2 - 2xy$
2. $f(x, y) = 3x + 4y^2$
3. $f(x, y) = \dfrac{y}{x} + \dfrac{x}{y}$
4. $f(x, y) = \ln(x^2 + y^2)$
5. $f(x, y) = xe^{3-y}$
6. $f(x, y) = e^{x+y}$
7. $f(x, y) = \sin(x + 2y)$
8. $f(x, y, z) = xyz^2$
9. $g(x, y, z) = xe^{y+3z}$
10. $f(x, y, z) = \dfrac{xy - 1}{z + x}$

Compute the directional derivative of the functions given in Problems 11–16 at the point P_0 in the direction of the given vector **v**.

Function	**Point P_0**	**Vector v**
11. $f(x, y) = x^2 + xy$	$(1, -2)$	$\mathbf{i} + \mathbf{j}$
12. $f(x, y) = \dfrac{e^{-x}}{y}$	$(2, -1)$	$-\mathbf{i} + \mathbf{j}$
13. $f(x, y) = \ln(x^2 + 3y)$	$(1, 1)$	$\mathbf{i} + \mathbf{j}$
14. $f(x, y) = \ln(3x + y^2)$	$(0, 1)$	$\mathbf{i} - \mathbf{j}$
15. $f(x, y) = \sec(xy - y^3)$	$(2, 0)$	$-\mathbf{i} - 3\mathbf{j}$
16. $f(x, y) = \sin xy$	$(\sqrt{\pi}, \sqrt{\pi})$	$3\pi\mathbf{i} - \pi\mathbf{j}$

Find a unit vector that is normal to each surface given in Problems 17–24 at the prescribed point, and the standard form of the equation of the tangent plane at that point.

17. $x^2 + y^2 + z^2 = 3$ at $(1, -1, 1)$
18. $x^4 + y^4 + z^4 = 3$ at $(1, -1 -1)$
19. $\cos z = \sin(x + y)$ at $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$
20. $\sin(x + y) + \tan(y + z) = 1$ at $(\frac{\pi}{4}, \frac{\pi}{4}, -\frac{\pi}{4})$
21. $\ln\left(\dfrac{x}{y - z}\right) = 0$ at $(2, 5, 3)$
22. $\ln\left(\dfrac{x - y}{y + z}\right) = x - z$ at $(1, 0, 1)$
23. $ze^{x+2y} = 3$ at $(2, -1, 3)$
24. $ze^{x^2-y^2} = 3$ at $(1, 1, 3)$

Find the direction from P_0 in which the given function f increases most rapidly and compute the magnitude of the greatest rate of increase in Problems 25–34.

25. $f(x, y) = 3x + 2y - 1$; $P_0(1, -1)$
26. $f(x, y) = 1 - x^2 - y^2$; $P_0(1, 2)$

27. $f(x, y)$
28. $f(x, y)$
29. $f(x, y,$ assume
30. $f(x, y)$
31. $f(x, y$
32. $f(x, y$
33. $f(x, y,$
34. $f(x, y$

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27. $f(x, y) = x^3 + y^3$; $P_0(3, -3)$

28. $f(x, y) = ax + by + c$; $P_0(a, b)$

29. $f(x, y, z) = ax^2 + by^2 + cz^2$; $P_0(a, b, c)$; assume $a^4 + b^4 + c^4 \neq 0$.

30. $f(x, y) = ax^3 + by^3$; $P_0(a, b)$; assume $a^6 + b^6 \neq 0$.

31. $f(x, y) = \ln\sqrt{x^2 + y^2}$; $P_0(1, 2)$

32. $f(x, y) = \sin xy$; $P_0\left(\frac{\sqrt{\pi}}{3}, \frac{\sqrt{\pi}}{2}\right)$

33. $f(x, y, z) = (x + y)^2 + (y + z)^2 + (x + z)^2$; $P_0(2, -1, 2)$

34. $f(x, y, z) = z \ln\left(\frac{y}{x}\right)$; $P_0(1, e, -1)$

In Problems 35–38, find a unit vector that is normal to the given graph at the point $P_0(x_0, y_0)$ on the graph. Assume that a, b, and c are constants.

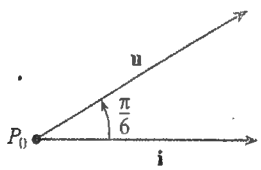
35. the line $ax + by = c$

36. the circle $x^2 + y^2 = a^2$

37. the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

38. the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

39. Find the directional derivative of $f(x, y) = x^2 + y^2$ at the point $P_0(1, 1)$ in the direction of the unit vector **u** shown in Figure 11.35.



$\hat{u} = \cos\frac{\pi}{6}\,\hat{i} + \sin\frac{\pi}{6}\,\hat{j}$

Figure 11.35 Problem 39

40. Find the directional derivative of $f(x, y) = x^2 + xy + y^2$ at $P_0(1, -1)$ in the direction toward the origin.

41. Find the directional derivative of $f(x, y) = e^{x^2 y^2}$ at $P_0(1, -1)$ in the direction toward $Q(2, 3)$.

42. Let $f(x, y, z) = 2x^2 - y^2 + 3z^2 - 8x - 4y + 201$, and let P_0 be the point $(2, -\frac{3}{2}, \frac{1}{2})$

a. Find ∇f_0.

b. Find $\cos\theta$, where θ is the angle between ∇f_0 and the vector toward the origin from P_0.

B 43. Let $f(x, y, z) = xyz$, and let **u** be a unit vector perpendicular to both $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$. Find the directional derivative of f at $P_0(1, -1, 2)$ in the direction of **u**.

44. Let $f(x, y, z) = ye^{x+z} + ze^{y-x}$. At the point $P(2, 2, -2)$, find the unit vector pointing in the direction of most rapid increase of f.

45. **Modeling Problem** Suppose a box in space given by $0 \le x \le 2$, $0 \le y \le 2$, $0 \le z \le 2$ is temperature-controlled so that the temperature at a point $P(x, y, z)$ in the box is modeled by $T(x, y, z) = xy + yz + xz$. A heat-seeking missile is located at $P_0(1, 1, 1)$. In what direction will the missile move for the temperature to increase as quickly as possible? What is the maximum rate of change of the temperature at the point P_0?

46. **Modeling Problem** A metal plate covering the rectangular region $0 < x \le 6$, $0 < y \le 5$ is charged electrically in such a way that the potential at each point (x, y) is inversely proportional to the square of its distance from the origin. If an object is at the point (3, 4), in which direction should it move to increase the potential most rapidly?

47. A hiker is walking on a mountain path when it begins to rain. If the surface of the mountain is modeled by $z = 1 - 3x^2 - \frac{5}{2}y^2$ (where x, y, and z are in miles) and the rain begins when the hiker is at the point $P_0\left(\frac{1}{4}, -\frac{1}{2}, \frac{3}{16}\right)$, in what direction should she head to descend the mountainside most rapidly?

48. Let f have continuous partial derivatives, and assume the maximal directional derivative of f at (0, 0) is equal to 100 and is attained in the direction toward (3, −4). Find the gradient ∇f at (0, 0).

49. Suppose at the point $P_0(-1, 2)$, a certain function $f(x, y)$ has directional derivative 8 in the direction of $\mathbf{v}_1 = 3\mathbf{i} - 4\mathbf{j}$ and 1 in the direction of $\mathbf{v}_2 = 12\mathbf{i} + 5\mathbf{j}$. What is the directional derivative of f at P_0 in the direction of $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j}$?

50. The directional derivative of $f(x, y, z)$ at the point P_0 is greatest in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and has value $5\sqrt{3}$ in this direction. What is the directional derivative of f at P_0 in the direction of $\mathbf{w} = \mathbf{i} + \mathbf{j}$?

51. Let f have continuous partial derivatives and suppose the maximal directional derivative of f at $P_0(1, 2)$ has magnitude 50 and is attained in the direction from P_0 toward $Q(3,-4)$. Use this information to find $\nabla f(1, 2)$.

52. Let $T(x, y) = 1 - x^2 - 2y^2$ be the temperature at each point $P(x, y)$ in the plane. A heat-loving bug is placed in the plane at the point $P_0(-1, 1)$. Find the path that the bug should take to stay as warm as possible. *Hint*: Assume that at each point on the bug's path, the tangent line will point in the direction in which T increases most rapidly.

53. a. Show that the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

has a tangent plane at $P_0(x_0, y_0, z_0)$ with the equation

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$$

b. Find the equation for the tangent plane to the hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

at the point $P_0(x_0, y_0, z_0)$.

c. Find the equation for the tangent plane to the elliptic paraboloid

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

at the point $P_0(x_0, y_0, z_0)$.

C 54. Recall that an ellipse is the set of all points $P(x, y)$ such that the sum of the distances from P to two fixed points (the *foci*) is constant. Let $P(x, y)$ be a point on the ellipse, and let r_1 and r_2 denote the respective distances from P to the two foci, F_1 and F_2 as shown in Figure 11.36.

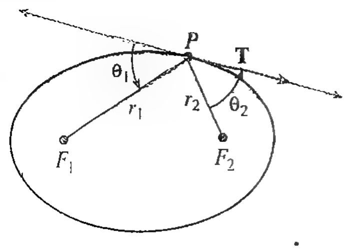


Figure 11.36 Unit tangent to an ellipse

a. Show that $\mathbf{T} \cdot \nabla(r_1 + r_2) = 0$, where $\mathbf{T}$ is a unit tangent to the ellipse at P.

b. Use part **a** to show that the tangent line to the ellipse at P makes equal angles with the lines joining P to the foci (that is, $\theta_1 = \theta_2$ in Figure 11.36).

55. **Modeling Problem** A particle P_1 with mass m_1 is located at the origin, and a particle P_2 with mass 1 unit is located at the point (x, y, z). According to Newton's law of universal gravitation, the force P_1 exerts on P_2 is modeled by

$$\mathbf{F} = \frac{-Gm_1(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{r^3}$$

where r is the distance between P_1 and P_2, and G is the gravitational constant.

a. Starting from the fact that $r^2 = x^2 + y^2 + z^2$, show that

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-x}{r^3}, \quad \frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \frac{-y}{r^3}, \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{-z}{r^3}$$

b. The function $V = -Gm_1/r$ is called the *potential energy* function for the system. Show that $\mathbf{F} = -\nabla V$.

56. Verify each of the following properties for functions of two variables.

a. $\nabla c = \mathbf{0}$ for constant c

b. $\nabla(f + g) = \nabla f + \nabla g$

c. $\nabla(f/g) = \dfrac{g\nabla f - f\nabla g}{g^2}$, $g \neq 0$

d. $\nabla(fg) = f\nabla g + g\nabla f$

57. Find the parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $2x^2 + 2y^2 + z^2 = 8$ at $P_0(-1, 1, 2)$.

58. Find a general formula for the directional derivative $D_{\mathbf{u}}f$ of the function $f(x, y)$ at the point $P(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$. Apply your formula to obtain the directional derivative of $f(x, y) = xy^2e^{x-2y}$ at $P_0(-1, 3)$ in the direction of the unit vector

$$\mathbf{u} = \left(\cos\tfrac{\pi}{6}\right)\mathbf{i} + \left(\sin\tfrac{\pi}{6}\right)\mathbf{j}$$

59. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are unit vectors and that f has continuous partial derivatives. Show that

$$D_{\mathbf{u}+\mathbf{v}}f = \frac{1}{\|\mathbf{u}+\mathbf{v}\|}(D_{\mathbf{u}}f + D_{\mathbf{v}}f)$$

60. **Counterexample Problem** If f, f_x, and f_y are continuous and $\nabla f(x, y) = \mathbf{0}$ inside the disk $x^2 + y^2 < 1$, then either show that $f(x, y)$ is a constant function inside the disk or find a counterexample.

61. **Counterexample Problem** If f is differentiable at $P_0(x_0, y_0)$ and $D_{\mathbf{u}}f(x_0, y_0) = 0$ for unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1 \times \mathbf{u}_2 \neq \mathbf{0}$, then show that $D_{\mathbf{u}}f(x_0, y_0) = 0$ for every unit vector $\mathbf{u}$, or find a counterexample.

62. Let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and let $r = \|\mathbf{R}\| = \sqrt{x^2 + y^2 + z^2}$.

a. Show that ∇r is a unit vector in the direction of $\mathbf{R}$.

b. Show that $\nabla(r^n) = nr^{n-2}\mathbf{R}$, for any positive integer n.

63. Suppose the surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ both pass through the point $P_0(x_0, y_0, z_0)$ and that the gradients ∇F_0 and ∇G_0 both exist. Show that the two surfaces are tangent at P_0 if and only if $\nabla F_0 \times \nabla G_0 = \mathbf{0}$.

11.7 Extrema of Functions of Two Variables

IN THIS SECTION relative extrema, second partials test, absolute extrema of continuous functions, least squares approximation of data

There are many practical situations in which it is necessary or useful to know the largest and smallest values of a function of two variables. For example, if $T(x, y)$ is the temperature at a point (x, y) in a plate, where are the hottest and coldest points in the plate and what are these extreme temperatures? If a hazardous waste dump is bounded by the curve $F(x, y) = 0$, what are the largest and smallest distances to the boundary from a given interior point P_0? We begin our study of extrema with some terminology.

Absolute Extrema

The function $f(x, y)$ is said to have an **absolute maximum** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in the domain D of f. Similarly, f has an **absolute minimum** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in D. Collectively, absolute maxima and minima are called **absolute extrema**.

Relati

In Chapter 4, we located absolute extrema of a function of one variable by first finding *relative extrema*, those values of $f(x)$ that are larger or smaller than those at all nearby points. The relative extrema of a function of two variables may be defined as follows.

Relative Extrema

> Let f be a function defined on a region containing (x_0, y_0). Then
>
> $f(x_0, y_0)$ is a **relative maximum** if $f(x, y) \le f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0).
>
> $f(x_0, y_0)$ is a **relative minimum** if $f(x, y) \ge f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0).
>
> Collectively, relative maxima and minima are called **relative extrema**.

RELATIVE EXTREMA

In Chapter 4, we observed that relative extrema of the function f correspond to "peaks and valleys" on the graph of f, and the same observation can be made about relative extrema in the two-variable case, as seen in Figure 11.37.

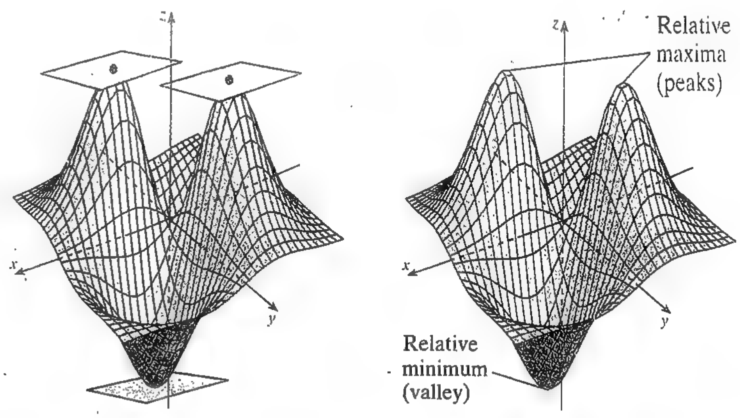


Figure 11.37 Relative extrema correspond to peaks and valleys

For a function f of one variable, we found that the relative extrema occur where $f'(x) = 0$ or $f'(x)$ does not exist. The following theorem shows that the relative extrema of a function of two variables can be located similarly.

THEOREM 11.11 Partial derivative criteria for relative extrema

If f has a relative extremum (maximum or minimum) at $P_0(x_0, y_0)$ and partial derivatives f_x and f_y both exist at (x_0, y_0), then

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

Proof Let $F(x) = f(x, y_0)$. Then $F(x)$ must have a relative extremum at $x = x_0$, so $F'(x_0) = 0$, which means that $f_x(x_0, y_0) = 0$. Similarly, $G(y) = f(x_0, y)$ has a relative extremum at $y = y_0$, so $G'(y_0) = 0$ and $f_y(x_0, y_0) = 0$. Thus, we must have both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, as claimed. ❑

WARNING There is a horizontal tangent plane at each extreme point where the first partial derivatives exist. However, this **does not** say that whenever a horizontal tangent plane occurs at a point P, there must be an extremum there. All that can be said is that such a point P is a possible location for a relative extremum.

In single-variable calculus, we referred to a number x_0, where $f'(x_0)$ does not exist or $f'(x_0) = 0$ as a *critical number*. This terminology is extended to functions of two variables as follows.

Critical Points

A **critical point** of a function f defined on an open set D is a point (x_0, y_0) in D where either one of the following is true:

a. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$
b. At least one of $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist at (x_0, y_0).

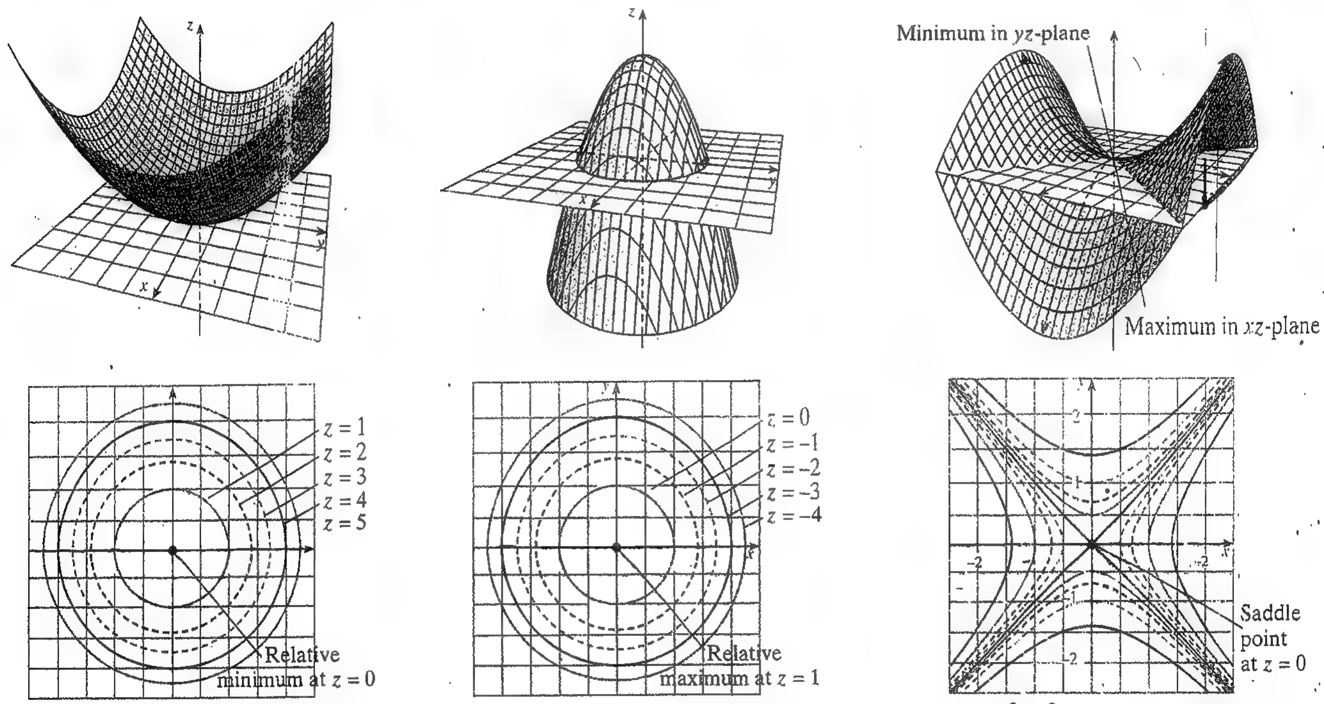
EXAMPLE 1 Distinguishing critical points

Discuss the nature of the critical point (0, 0) for the quadric surfaces

a. $z = x^2 + y^2$ **b.** $z = 1 - x^2 - y^2$ **c.** $z = y^2 - x^2$

Solution

The graphs of these quadric surfaces are shown in Figure 11.38. Let $f(x, y) = x^2 + y^2$, $g(x, y) = 1 - x^2 - y^2$, and $h(x, y) = y^2 - x^2$. We find the critical points:



a. $z = x^2 + y^2$ (paraboloid) and level curves; relative minimum at (0, 0)

b. $z = 1 - x^2 - y^2$ (paraboloid) and level curves; relative maximum at (0, 0)

c. $z = y^2 - x^2$ (hyperbolic paraboloid) and level curves; saddle point at (0, 0)

Figure 11.38 Classification of critical points

a. $f_x(x, y) = 2x$, $f_y(x, y) = 2y$; critical point (0, 0). The function f has a relative minimum at (0, 0) because x^2 and y^2 are both nonnegative, yielding $x^2 + y^2 > 0$ for all nonzero x and y.

b. $g_x(x, y) = -2x$, $g_y(x, y) = -2y$; critical point (0, 0). Since $z = 1 - x^2 - y^2$, it follows that $z \le 1$ with a relative maximum occurring where x^2 and y^2 are both 0; that is, at (0, 0).

c. $h_x(x, y) = -2x$, $h_y(x, y) = 2y$; critical point (0, 0). The function h has neither a relative maximum nor a relative minimum at (0, 0). When $z = 0$, h is a minimum on the y-axis (where $x = 0$) and a maximum on the x-axis (where $y = 0$). ■

A critical point $P_0(x_0, y_0)$ is called a **saddle point** of $f(x, y)$ if every open disk centered at P_0 contains points in the domain of f that satisfy $f(x, y) > f(x_0, y_0)$ as well as points in the domain of f that satisfy $f(x, y) < f(x_0, y_0)$. An example of a saddle point is (0, 0) on the hyperbolic paraboloid $z = y^2 - x^2$, as shown in Figure 11.38c.

SECOND PARTIALS TEST

The previous example points to the need for some sort of a test to determine the nature of a critical point. In Chapter 4, we developed the second derivative test for functions of one variable as a means for determining whether a particular critical number c of f corresponds to a relative maximum or minimum. If $f'(c) = 0$, then according to this test, a relative maximum occurs at $x = c$ if $f''(c) < 0$ and a relative minimum occurs if $f''(c) > 0$. If $f''(c) = 0$, the test is inconclusive. The analogous result for the two-variable case may be stated as follows.

THEOREM 11.12 **Second partials test**

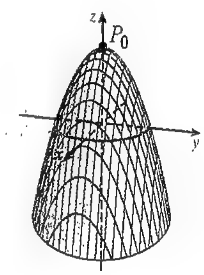
Let $f(x, y)$ have a critical point at $P_0(x_0, y_0)$ and assume that f has continuous second-order partial derivatives in a disk centered at (x_0, y_0). The *discriminant of f* is the expression

$$\boxed{D = f_{xx}f_{yy} - f_{xy}^2}$$

Then,

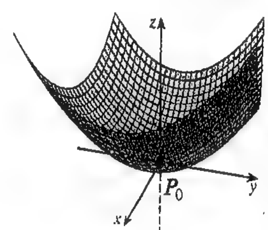
A **relative maximum** occurs at P_0 if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$ (or, equivalently, $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0$).

$z = 1 - x^2 - y^2$ (See Ex 1b)



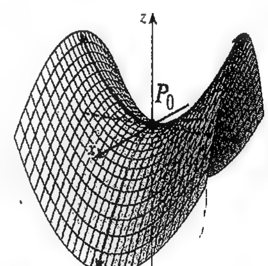
A **relative minimum** occurs at P_0 if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$ (or $f_{yy}(x_0, y_0) > 0$).

$z = x^2 + y^2$ (See Ex 1a)



A **saddle point** occurs at P_0 if $D(x_0, y_0) < 0$.

$z = y^2 - x^2$ (See Ex 1c)



If $D(x_0, y_0) = 0$, then the test is **inconclusive**. Nothing can be said about the nature of the surface at (x_0, y_0) without further analysis.

Proof The second partials test can be proved by using an extension of the Taylor series expansion (Section 8.8) that applies to functions of two variables $f(x, y)$. Details can be found in most advanced calculus texts (also see Problem 60). ❑

The discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ may be easier to remember in the equivalent determinant form

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Summary: For a critical point (a, b):

$D(a, b)$	$f_{xx}(a, b)$	**Type**
+	−	Rel. max.
+	+	Rel. min.
−	NA	Saddle point
0	NA	Inconclusive

SMH The definition of a 2×2 determinant is presented in Section 2.8 of the *Student Mathematics Handbook.*

Note that if $D > 0$ at the critical point $P_0(x_0, y_0)$, then f_{xx} and f_{yy} must have the same sign. This is the reason that when $D > 0$, either $f_{xx} > 0$ or $f_{yy} > 0$ is enough to guarantee that a relative minimum occurs at P_0 (or a relative maximum if $f_{xx} < 0$ or $f_{yy} < 0$).

Geometrically, if $D > 0$ and $f_{xx} > 0$ and $f_{yy} > 0$ at P_0, then the surface $z = f(x, y)$ curves upward in all directions from the point $Q(x_0, y_0, z_0)$, so there is a relative minimum at Q. Likewise, if $D > 0$ and $f_{xx} < 0$ and $f_{yy} < 0$ at P_0, then the surface curves downward in all directions from P_0, which must therefore be a relative maximum. However, if $D < 0$ at P_0, the surface curves up from Q in some directions, and down in others, so Q must be a saddle point.

WARNING The second partials test says nothing about the geometry of the surface at Q if $D = 0$ at P_0. Example 4 and Problem 54 show that a relative minimum, a relative maximum, a saddle point, or something entirely different may occur if $D = 0$.

EXAMPLE 2 Using the second partials test to classify critical points

Find all relative extrema and saddle points of the function

$$f(x, y) = 2x^2 + 2xy + y^2 - 2x - 2y + 5$$

Solution

First, find the critical points:

$$f_x = 4x + 2y - 2 \qquad f_y = 2x + 2y - 2$$

Setting $f_x = 0$ and $f_y = 0$, we obtain the system of equations

$$\begin{cases} 4x + 2y - 2 = 0 \\ 2x + 2y - 2 = 0 \end{cases}$$

and solve to obtain $x = 0$, $y = 1$. Thus, $(0, 1)$ is the only critical point. To apply the second partials test, we obtain

$$f_{xx} = 4 \quad f_{yy} = 2 \quad f_{xy} = 2$$

and form the discriminant

$$D = f_{xx}f_{yy} - f_{xy}^2 = (4)(2) - 2^2 = 4$$

For the critical point $(0, 1)$ we have $D = 4 > 0$ and $f_{xx} = 4 > 0$, so there is a relative minimum at $(0, 1)$. ■

EXAMPLE 3 Second partials test with a relative minimum and a saddle point

Find all critical points on the graph of $f(x, y) = 8x^3 - 24xy + y^3$, and use the second partials test to classify each point as a relative extremum or a saddle point.

Solution

$$f_x(x, y) = 24x^2 - 24y, \quad f_y(x, y) = -24x + 3y^2$$

To find the critical points, solve

$$\begin{cases} 24x^2 - 24y = 0 \\ -24x + 3y^2 = 0 \end{cases}$$

From the first equation, $y = x^2$; substitute this into the second equation to find

$$-24x + 3(x^2)^2 = 0$$
$$x(x^3 - 8) = 0$$
$$x(x-2)(x^2+2x+4) = 0$$
$$x = 0, 2 \qquad \text{The solutions of } x^2 + 2x + 4 = 0 \text{ are not real.}$$

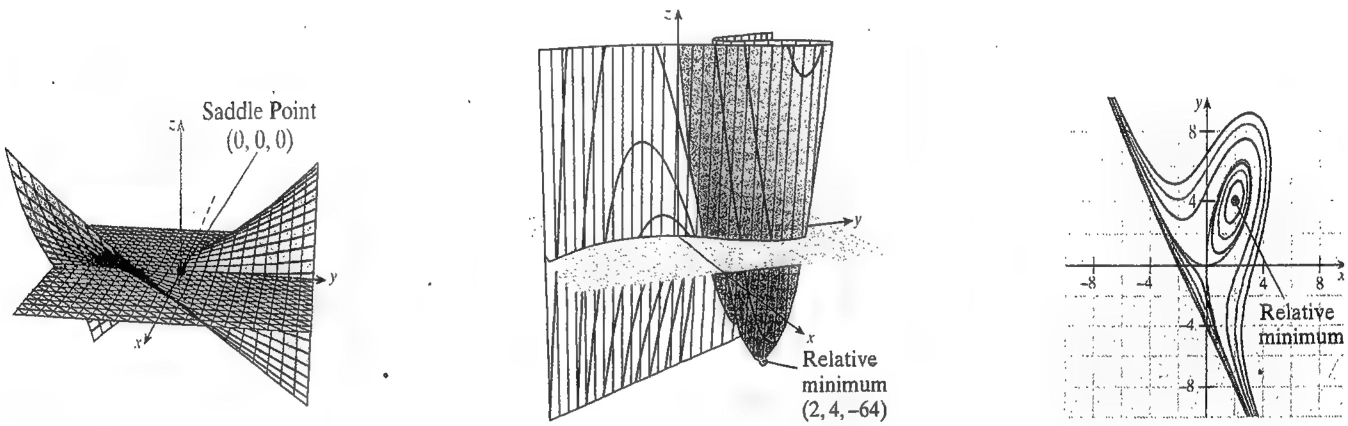
If $x = 0$, then $y = 0$, and if $x = 2$, then $y = 4$, so the critical points are (0, 0) and (2, 4). To obtain D, we first find $f_{xx}(x, y) = 48x$, $f_{xy}(x, y) = -24$, and $f_{yy}(x, y) = 6y$ to find $D(x, y) = (48x)(6y) - (-24)^2$ and then compute:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 48x & -24 \\ -24 & 6y \end{vmatrix} = 288xy - 576$$

Critical point (a, b)	$D(a, b)$	$f_{xx}(a, b)$	type
(0, 0)	Neg.		Saddle
(2, 4)	Pos.	Pos.	Rel. min.

At (0, 0), $D = -576 < 0$, so there is a saddle point at (0, 0).
At (2, 4), $D = 288(2)(4) - 576 = 1{,}728 > 0$ and $f_{xx}(2, 4) = 96 > 0$, so there is a relative minimum at (2, 4).

To view the situation graphically, we calculate the coordinates of the saddle point (0, 0, 0), and the relative minimum (2, 4, −64), as shown in Figure 11.39.



a. View near the origin (showing the saddle point)

b. View away from the origin (showing the relative minimum point)

c. Level curves

Figure 11.39 Graph of $f(x, y) = 8x^3 - 24xy + y^3$ ■

EXAMPLE 4 Extrema when the second partials test fails

Find all relative extrema and saddle points on the graph of

$$f(x, y) = x^2y^4$$

The graph is shown in Figure 11.40.

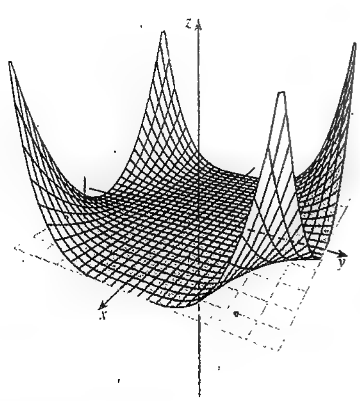


Figure 11.40 Graph of $f(x, y) = x^2y^4$

Solution

Since $f_x(x, y) = 2xy^4$, $f_y(x, y) = 4x^2y^3$, we see that the critical points occur only when $x = 0$ or $y = 0$; that is, every point on the x-axis or y-axis is a critical point. Because

$$f_{xx}(x, y) = 2y^4, \quad f_{xy}(x, y) = 8xy^3, \quad f_{yy}(x, y) = 12x^2y^2$$

the discriminant is

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2y^4 & 8xy^3 \\ 8xy^3 & 12x^2y^2 \end{vmatrix} = 24x^2y^6 - 64x^2y^6 = -40x^2y^6$$

Since $D = 0$ for any critical point $(x_0, 0)$ or $(0, y_0)$, the second partials test fails. However, $f(x, y) = 0$ for every critical point (because either $x = 0$ or $y = 0$, or both), and because $f(x, y) = x^2y^4 > 0$ when $x \neq 0$ and $y \neq 0$, it follows that each critical point must be a relative minimum. ■

ABSOLUTE EXTREMA OF CONTINUOUS FUNCTIONS

The extreme value theorem (Theorem 4.1) says that a function of a single variable f must attain both an absolute maximum and an absolute minimum on any closed, bounded interval $[a, b]$ on which it is continuous. In $\mathbb{R}^2$, a nonempty set S is **closed** if it contains its boundary (see the introduction to Section 11.2) and is **bounded** if it is contained in a disk. The extreme value theorem can be extended to functions of two variables in the following form.

THEOREM 11.13 Extreme value theorem for a function of two variables
A function of two variables $f(x, y)$ attains both an absolute maximum and an absolute minimum on any closed, bounded set S where it is continuous.

Proof The proof is found in most advanced calculus texts. ❑

To find the absolute extrema of a continuous function $f(x, y)$ on a closed, bounded set S, we proceed as follows:

Procedure for Determining Absolute Extrema

Given a function f that is continuous on a closed, bounded set S.

Step 1. Find all critical points of f in S.
Step 2. Find all points on the boundary of S where absolute extrema can occur.
Step 3. Compute the value of $f(x_0, y_0)$ for each of the points (x_0, y_0) found in steps 1 and 2.

Evaluation: The absolute maximum of f on S is the largest of the values computed in step 3, and the absolute minimum is the smallest of the computed values.

EXAMPLE 5 Finding absolute extrema

Find the absolute extrema of the function $f(x, y) = e^{x^2-y^2}$ over the disk $x^2 + y^2 \leq 1$. The graph is shown in Figure 11.41.

Figure 11.41 Graph of $f(x, y) = e^{x^2-y^2}$ over the disk $x^2 + y^2 \leq 1$

Solution

Step 1. $f_x(x, y) = 2xe^{x^2-y^2}$ and $f_y(x, y) = -2ye^{x^2-y^2}$. These partial derivatives are defined for all (x, y). Because $f_x(x, y) = f_y(x, y) = 0$ only when $x = 0$ and $y = 0$, it follows that $(0, 0)$ is the only critical point of f and it is inside the disk.

Step 2. Examine the values of f on the boundary curve $x^2 + y^2 = 1$. Because $y^2 = 1 - x^2$ on the boundary of the disk, we find that

$$f(x, y) = e^{x^2-(1-x^2)} = e^{2x^2-1}$$

We need to find the largest and smallest values of $F(x) = e^{2x^2-1}$ for $-1 \leq x \leq 1$. Since

$$F'(x) = 4xe^{2x^2-1}$$

we see that $F'(x) = 0$ only when $x = 0$ (since e^{2x^2-1} is always positive). At $x = 0$, we have $y^2 = 1 - 0^2$, so $y = \pm 1$; thus $(0, 1)$ and $(0, -1)$ are boundary critical points. At the endpoints of the interval $-1 \leq x \leq 1$, the corresponding points are $(1, 0)$ and $(-1, 0)$.

Step 3. Compute the value of f for the points found in steps 1 and 2:

Points to check	Compute $f(x_0, y_0) = e^{x_0^2 - y_0^2}$
$(0, 0)$	$f(0, 0) = e^0 = 1$
$(0, 1)$	$f(0, 1) = e^{-1}$; minimum
$(0, -1)$	$f(0, -1) = e^{-1}$; minimum
$(1, 0)$	$f(1, 0) = e$; maximum
$(-1, 0)$	$f(-1, 0) = e$; maximum

Evaluation: As indicated in the preceding table, the absolute maximum value of f on the given disk is e, which occurs at $(1, 0)$ and $(-1, 0)$, and the absolute minimum value is e^{-1}, which occurs at $(0, 1)$ and $(0, -1)$. ■

In general, it can be difficult to show that a relative extremum is actually an absolute extremum. In practice, however, it is often possible to make the determination using physical or geometric considerations. Consider the following example.

EXAMPLE 6 Minimum distance from a point to a plane

Find the point on the plane $x + 2y + z = 5$ that is closest to the point $P(0, 3, 4)$.

Solution

If $Q(x, y, z)$ is a point on the plane $x + 2y + z = 5$, then $z = 5 - x - 2y$ and the distance from P to Q is

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-3)^2 + (z-4)^2} \\ &= \sqrt{x^2 + (y-3)^2 + (5-x-2y-4)^2} \end{aligned}$$

Instead of minimizing d, we minimize

$$f(x, y) = d^2 = x^2 + (y-3)^2 + (1-x-2y)^2$$

since the minimum of d will occur at the same points where d^2 is also minimized.

To minimize $f(x, y)$, we first determine the critical points of f by solving the system

$$\begin{aligned} f_x &= 2x - 2(1-x-2y) = 4x + 4y - 2 = 0 \\ f_y &= 2(y-3) - 4(1-x-2y) = 4x + 10y - 10 = 0 \end{aligned}$$

We obtain $x = -\frac{5}{6}$, $y = \frac{4}{3}$, and since

$$f_{xx} = 4, \qquad f_{yy} = 10, \qquad f_{xy} = 4$$

we find that

$$D = f_{xx}f_{yy} - f_{xy}^2 = 4(10) - 4^2 > 0 \quad \text{and} \quad f_{xx} = 4 > 0$$

so a relative minimum occurs at $\left(-\frac{5}{6}, \frac{4}{3}\right)$.

Intuitively, we see that this relative minimum must also be an absolute minimum because there must be exactly one point on the plane that is closest to the given point. The corresponding z-value is $z = 5 - \left(-\frac{5}{6}\right) - 2\left(\frac{4}{3}\right) = \frac{19}{6}$. Thus, the closest point on the plane is $Q\left(-\frac{5}{6}, \frac{4}{3}, \frac{19}{6}\right)$, and the minimum distance is

$$d = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{4}{3} - 3\right)^2 + \left[1 + \frac{5}{6} - 2\left(\frac{4}{3}\right)\right]^2} = \sqrt{\frac{25}{6}} = \frac{5}{\sqrt{6}}$$

Check: You might want to check your work by using the formula for the distance from a point to a plane in $\mathbb{R}^3$ (Theorem 9.8):

$$d = \left|\frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}\right| = \left|\frac{0 + 2(3) + 4 - 5}{\sqrt{1^2 + 2^2 + 1^2}}\right| = \frac{5}{\sqrt{6}}$$

■

LEAST SQUARES APPROXIMATION OF DATA

In the following example, calculus is applied to justify a formula used in statistics and in many applications in the social and physical sciences.

don't do

EXAMPLE 7 Least squares approximation of data

Suppose data consisting of n points $P_1, \ldots, P_n$ are known, and we wish to find a function $y = f(x)$ that fits the data reasonably well. In particular, suppose we wish to find a line $y = mx + b$ that "best fits" the data in the sense that the sum of the squares of the vertical distances from each data point to the line is minimized.

Solution

We wish to find values of m and b that minimize the sum of the squares of the differences between the y-values and the line $y = mx + b$. The line that we seek is called the **regression line**. Suppose that the point P_k has components (x_k, y_k). At this point the value on the regression line is $y = mx_k + b$ and the value of the data point is y_k. The "error" caused by using the point on the regression line rather than the actual data point can be measured by the difference

$$y_k - (mx_k + b)$$

The data points may be above the regression line for some values of k and below the regression line for other values of k. We see that we need to minimize the function that represents the sum of the *squares* of all these differences:

$$F(m, b) = \sum_{k=1}^{n} [y_k - (mx_k + b)]^2$$

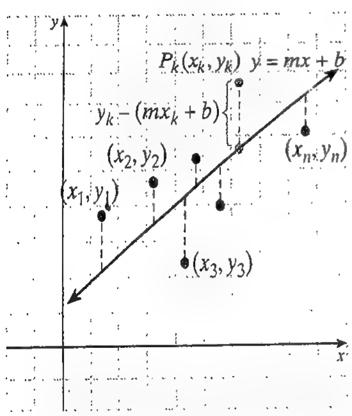


Figure 11.42 Least squares approximation of data

The situation is illustrated in Figure 11.42. To find where F is minimized, we first compute the partial derivatives.

$$\begin{aligned} F_m(m, b) &= \sum_{k=1}^{n} 2[y_k - (mx_k + b)](-x_k) \\ &= 2m\sum_{k=1}^{n} x_k^2 + 2b\sum_{k=1}^{n} x_k - 2\sum_{k=1}^{n} x_k y_k \\ F_b(m, b) &= \sum_{k=1}^{n} 2[y_k - (mx_k + b)](-1) \\ &= 2m\sum_{k=1}^{n} x_k + 2b\sum_{k=1}^{n} 1 - 2\sum_{k=1}^{n} y_k \\ &= 2m\sum_{k=1}^{n} x_k + 2bn - 2\sum_{k=1}^{n} y_k \end{aligned}$$

Set each of these partial derivatives equal to 0 to find the critical values (see Problem 61).

$$m = \frac{n\displaystyle\sum_{k=1}^{n} x_k y_k - \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right)}{n\displaystyle\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2} \quad \text{and} \quad b = \frac{\displaystyle\sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k - \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k y_k\right)}{n\displaystyle\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2}$$

It can be shown that these values of m and b yield an absolute minimum for $F(m, b)$. ■

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Most applications of the **least squares formula** stated in Example 7 involve using a calculator or computer software. The following technology note provides an example.

TECHNOLOGY NOTE

Many calculators will carry out the calculations required by the least squares approximation procedure. Look at your owner's manual for specifics. Most calculators allow you to input data with keys labeled $\boxed{\text{STAT}}$ and $\boxed{\text{DATA}}$. After the data are input, the m and b values are given by pressing the $\boxed{\text{LinReg}}$ choice. For example, ten people are given a standard IQ test. Their scores are then compared with their high school grades:

IQ:	117	105	111	96	135	81	103	99	107	109
GPA:	3.1	2.8	2.5	2.8	3.4	1.9	2.1	3.2	2.9	2.3

A calculator output shows: $m = .0224144711$ and $b = .3173417224$. A scatter diagram with the least squares line is shown in Figure 11.43.

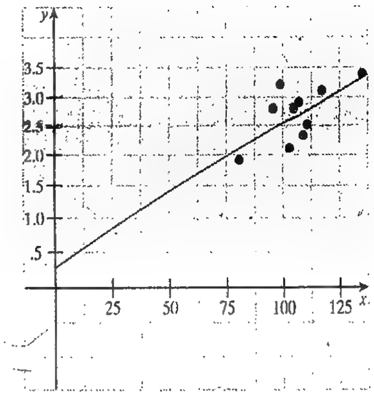

Figure 11.43 Scatter diagram and least squares line

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Guidelines 8. 1–30

11.7 PROBLEM SET

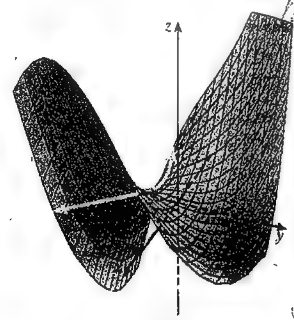
A 1. **WHAT DOES THIS SAY?** Describe what is meant by a critical point.

2. **WHAT DOES THIS SAY?** Describe a procedure for classifying relative extrema.

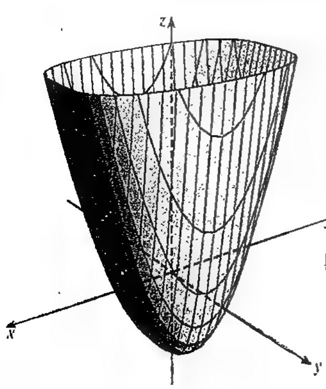
3. **WHAT DOES THIS SAY?** Describe a procedure for determining absolute extrema on a closed, bounded set S.

Find the critical points in Problems 4–23, and classify each point as a relative maximum, a relative minimum, or a saddle point.

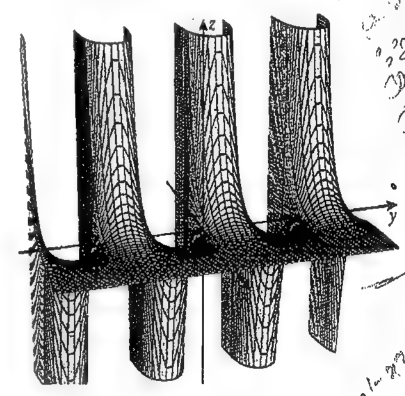
4. $f(x, y) = 2x^2 - 4xy + y^3 + 2$



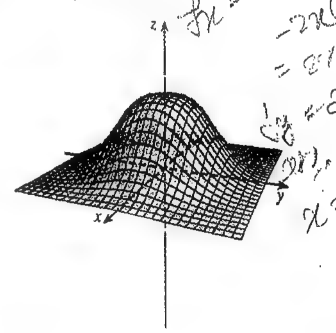
5. $f(x, y) = (x - 2)^2 + (y - 3)^4$



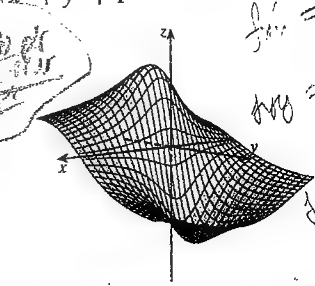
6. $f(x, y) = e^{-x}\sin y$



7. $f(x, y) = (1 + x^2 + y^2)e^{1-x^2-y^2}$



8. $f(x, y) = \dfrac{9x}{x^2 + y^2 + 1}$



9. $f(x, y) = x^2 + xy + y^2$
10. $f(x, y) = xy - x + y$
11. $f(x, y) = -x^3 + 9x - 4y^2$
12. $f(x, y) = e^{-(x^2+y^2)}$
13. $f(x, y) = (x^2 + 2y^2)e^{1-x^2-y^2}$
14. $f(x, y) = e^{xy}$
15. $f(x, y) = x^{-1} + y^{-1} + 2xy$
16. $f(x, y) = (x - 4)\ln(xy)$
17. $f(x, y) = x^3 + y^3 + 3x^2 - 18y^2 + 81y + 5$
18. $f(x, y) = 2x^3 + y^3 + 3x^2 - 3y - 12x - 4$
19. $f(x, y) = x^2 + y^2 - 6xy + 9x + 5y + 2$
20. $f(x, y) = x^2 + y^2 + \dfrac{32}{xy}$
21. $f(x, y) = x^2 + y^3 + \dfrac{768}{x+y}$
22. $f(x, y) = 3xy^2 - 2x^2y + 36xy$
23. $f(x, y) = 3x^2 + 12x + 8y^3 - 12y^2 + 7$

B *Find the absolute extrema of f on the closed bounded set S in the plane as described in Problems 24–30.*

24. $f(x, y) = 2x^2 - y^2$; S is the disk $x^2 + y^2 \leq 1$.
25. $f(x, y) = xy - 2x - 5y$; S is the triangular region with vertices $(0, 0)$, $(7, 0)$, and $(7, 7)$.
26. $f(x, y) = x^2 + 3y^2 - 4x + 2y - 3$; S is the square region with vertices $(0, 0)$, $(3, 0)$, $(3, -3)$, and $(0, -3)$.
27. $f(x, y) = 2\sin x + 5\cos y$; S is the rectangular region with vertices $(0, 0)$, $(2, 0)$, $(2, 5)$, and $(0, 5)$.
28. $f(x, y) = e^{x^2+2x+y^2}$; S is the disk $x^2 + 2x + y^2 \leq 0$.
29. $f(x, y) = x^2 + xy + y^2$; S is the disk $x^2 + y^2 \leq 1$.
30. $f(x, y) = x^2 - 4xy + y^3 + 4y$; S is the square region $0 \leq x \leq 2$, $0 \leq y \leq 2$.

Find the least squares regression line for the data points given in Problems 31–34.

31. $(-2,-3)$, $(-1,-1)$, $(0, 1)$, $(1, 3)$, $(3, 5)$
32. $(0, 1)$, $(1, 1.6)$, $(2.2, 3)$, $(3.1, 3.9)$, $(4, 5)$
33. $(3, 5.72)$, $(4, 5.31)$, $(6.2, 5.12)$, $(7.52, 5.32)$, $(8.03, 5.67)$
34. $(-4, 2)$, $(-3, 1)$, $(0, 0)$, $(1, -3)$, $(2, -1)$, $(3, -2)$
35. Find all points on the surface $y^2 = 4 + xz$ that are closest to the origin.
36. Find all points in the plane $x + 2y + 3z = 4$ in the first octant where $f(x, y, z) = x^2yz^3$ has a maximum value.
37. A rectangular box with no top is to have a fixed volume. What should its dimensions be if we want to use the least amount of material in its construction?
38. A wire of length L is cut into three pieces that are bent to form a circle, a square, and an equilateral triangle. How should the cuts be made to minimize the sum of the total area?
39. Find three positive numbers whose sum is 54 and whose product is as large as possible.
40. A dairy produces whole milk and skim milk in quantities x and y pints, respectively. Suppose the price (in cents) of whole milk is $p(x) = 100 - x$ and that of skim milk is $q(y) = 100 - y$, and also assume that $C(x, y) = x^2 + xy + y^2$ is the joint-cost function of the commodities. Maximize the profit

$$P(x, y) = px + qy - C(x, y)$$

41. Let R be the triangular region in the xy-plane with vertices $(-1, -2)$, $(-1, 2)$, and $(3, 2)$. A plate in the shape of R is heated so that the temperature at (x, y) is

$$T(x, y) = 2x^2 - xy + y^2 - 2y + 1$$

(in degrees Celsius). At what point in R or on its boundary is T maximized? Where is T minimized? What are the extreme temperatures?

42. A particle of mass m in a rectangular box with dimensions x, y, z has *ground state energy*

$$E(x, y, z) = \frac{k^2}{8m}\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$$

where k is a physical constant. If the volume of the box is fixed (say $V_0 = xyz$), find the values of x, y, and z that minimize the ground state energy.

43. A manufacturer produces two different kinds of graphing calculators, A and B, in quantities x and y (units of 1,000), respectively. If the revenue function (in dollars) is $R(x, y) = -x^2 - 2y^2 + 2xy + 8x + 5y$, find the quantities of A and B that should be produced to maximize revenue.
44. Suppose we wish to construct a closed rectangular box with volume 32 ft^3. Three different materials will be used in the construction. The material for the sides costs \$1 per square foot, the material for the bottom costs \$3 per square foot, and the material for the top costs \$5 per square foot. What are the dimensions of the least expensive such box?
45. **Modeling Problem** A store carries two competing brands of bottled water, one from California and the other from upstate, New York. To model this situation, assume the owner of the store can obtain both at a cost of \$2/bottle. Also assume that if the California water is sold for x dollars per bottle and the New York water for y dollars per bottle, then consumers will buy approximately $40 - 50x + 40y$ bottles of California water and $20 + 60x - 70y$ bottles of the New York water each day. How should the owner price the bottled water to generate the largest possible profit?
46. **Modeling Problem** A telephone company is planning to introduce two new types of executive communication systems that it hopes to sell to its largest commercial customers. To create a model to determine the maximum profit, it is assumed that if the first type of system is priced at x hundred dollars per system and the second type at y hundred dollars per system, approximately $40 - 8x + 5y$ consumers will buy the first type and $50 + 9x - 7y$ will buy the second type. If the cost of manufacturing the first type is \$1,000 per system and the cost of manufacturing the second type is \$3,000 per system, how should the telephone company price the systems to generate maximum profit?
47. **Modeling Problem** A manufacturer with exclusive rights to a sophisticated new industrial machine is planning to sell a limited number of the machines to both foreign and domestic firms. The price the manufacturer can expect to receive for the machines will depend on the number of machines made available. For example, if only a few of the machines are placed on the market, competitive bidding among prospective purchasers will tend to drive the price up. It is estimated that if the manufacturer supplies x machines to the domestic market and y machines to the foreign market, the machines will sell for $60 - 0.2x + 0.05y$ thousand dollars apiece at home and $50 + 0.1y + 0.05x$ thousand dollars apiece abroad. If the manufacturer can produce the machines at a total cost of \$10,000 apiece, how many should be supplied to each market to generate the largest possible profit?

48. A college admissions officer, Dr. Westfall, has compiled the following data relating students' high-school and college GPAs:

HS GPA	2.0	2.5	3.0	3.0	3.5	3.5	4.0	4.0
College GPA	1.5	2.0	2.5	3.5	2.5	3.0	3.0	3.5

Plot the data points on a graph and find the equation of the regression line for these data. Then use the regression line to predict the college GPA of a student whose high school GPA is 3.75.

49. It is known that if an ideal spring is displaced a distance y from its natural length by a force (weight) x, then $y = kx$, where k is the so-called spring constant. To compute this constant for a particular spring, a scientist obtains the following data:

x (lb)	5.2	7.3	8.4	10.12	12.37
y (in.)	11.32	15.56	17.44	21.96	26.17

Based on these data, what is the "best" choice for k?

50. **Exploration Problem** The following table gives the approximate U.S. census figures (in millions):

Year:	1900	1910	1920	1930	1940
Population:	76.2	92.2	106.0	123.2	132.1

Year:	1950	1960	1970	1980	1990
Population:	151.3	179.3	203.3	226.5	248.7

a. Find the least squares regression line for the given data and use this line to "predict" the population in 1997. (The actual population was about 266.5 million.)

b. Use the least squares linear approximation to estimate the population at the present time. Check your answer by looking up the population using the Internet. Comment on the accuracy (or inaccuracy) of your prediction.

51. **Exploration Problem** The following table gives the Dow Jones Industrial Average (DJIA) Stock Index along with the per capita consumption of wine (in gallons) for those years.

Year:	1965	1970	1975	1980	1985	1990	1995
DJIA Index:	911	753	802	891	1,328	2,796	3,838
Consumption:	0.98	1.31	1.71	2.11	2.43	2.05	1.79

a. Plot these data on a graph, with the DJIA Index on the x-axis and consumption on the y-axis.

b. Find the equation of the least squares line.

c. Determine whether the consumption figures predicted by the least squares line in part **b** are approximately correct by using the most recent figures available at the time of this writing: The DJIA opened 1997 at 6,442 and U.S. wine consumption was 1.95 gallons. Interpret your findings.

d. In 2001 the stock market began the year at 10,790. Use the least squares line to predict per capita wine consumption that corresponds to this stock value.

52. **Linearizing nonlinear data** In this problem, we turn to data that do *not* tend to change linearly. Often we can "linearize" the data by taking the logarithm or exponential of the data and then doing a linear fit as described in this section.

a. Suppose we have, or suspect, a relationship $y = kx^m$. Show that by taking the natural logarithm of this equation we obtain a linear relationship: $Y = K + mX$. Explain the new variables and constant K.

b. Following are data relating the periods of revolution t (in days) of the six inner planets and their semimajor axis a (in 10^6 km). Kepler conjectured the relationship $t = ka^m$, which is very accurate for the correct k and a. You are to "transform" the data as in part **a** (thus obtaining $T_i = \ln t_i, \ldots$) and do a linear fit to the new data, thus finding k and m.

t-data: (87.97, 224.7, 365.26, 686.98, 4332.59, 10759.2)
a-data: (58, 108, 149, 228, 778, 1426)

53. Following are data pertaining to a recent Olympic weight-lifting competition. The x-data are the "class data" giving eight weight classes (in kg) from featherweight to heavyweight-2. The w-data are the combined weights lifted by the winners in each class. Theoretically, we would expect a relationship $w = kx^m$, where $m = 2/3$. (Can you see why?)

a. Linearize the data as in Problem 52 and use the least squares approximation to find k and m.

x-data: (56, 60, 67.5, 75, 82.5, 90, 100, 110)
w-data: (292.5, 342.5, 340, 375, 377.5, 412.5, 425, 455)

b. Comment on the 60-kg entry. Do you see why this participant (N. Suleymanoglu of Turkey) was referred to as the strongest man in the world?

54. This problem is designed to show, by example, that if $D = 0$ at a critical point, then almost anything can happen.

a. Show that $f(x, y) = x^4 - y^4$ has a saddle point at (0, 0).

b. Show that $g(x, y) = x^2 y^2$ has a relative minimum at (0, 0).

c. Show that $h(x, y) = x^3 + y^3$ has no extremum or saddle point at (0, 0).

55. **Counterexample Problem** If f is a continuous function of one variable with two relative maxima on a given interval, there must be a relative minimum between the maxima. By considering the function*

$$f(x, y) = 4x^2 e^y - 2x^4 - e^{4y}$$

show that it is possible for a continuous function of two variables to have only two relative maxima and no relative minima.

56. **Counterexample Problem** If a continuous function of one variable has only one critical number on an interval, then a relative extremum must also be an absolute extremum. Show that this result does not extend to functions of two variables by considering the function†

$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

In particular, show that it has exactly one critical point, which corresponds to a relative maximum. Is this also an absolute maximum? Explain.

57. Tom, Dick, and Mary are participating in a cross-country relay race. Tom will trudge as fast as he can through thick woods to the edge of a river, and then Dick will take over and row to the

*Based on the problem "Two Mountains Without a Valley," by Ira Rosenholtz, *Mathematics Magazine*, 1987, Vol. 60, No. 1, p. 48.

†Based on material in the article, "The Only Critical Point in Town Test" by Ira Rosenholtz and Lowell Smythe, *Mathematics Magazine*, 1985, Vol. 58, No. 3, pp. 149–150.

opposite shore. Finally, Mary will take the baton and run along the river road to the finish line. The course is shown in Figure 11.44.

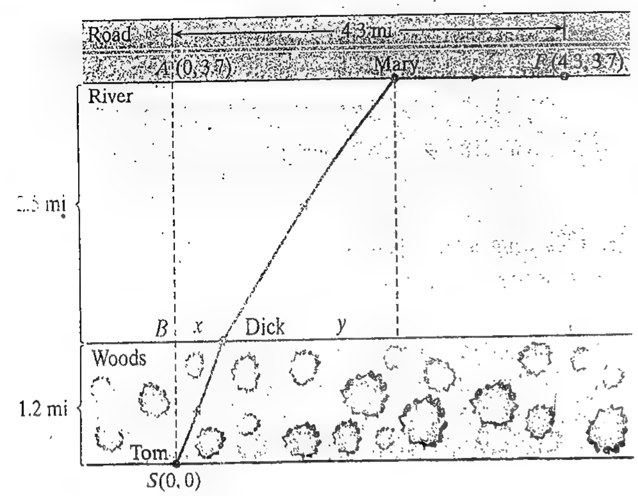


Figure 11.44 Course for the relay

Teams must start at point S and finish at point F, but they may position one member anywhere along the shore of the river and another anywhere along the river road. Suppose Tom can trudge at 2 mi/h, Dick can row at 4 mi/h, and Mary can run at 6 mi/h. Where should Dick and Mary wait to receive the baton in order for the team to finish the course as quickly as possible?

58. Exploration Problem Consider the function $f(x, y) = (y - x^2)(y - 2x^2)$. Discuss the behavior of this function at $(0, 0)$.

59. Exploration Problem Sometimes the critical points of a function can be classified by looking at the level curves. In each case shown in Figure 11.45, determine the nature of the critical point(s) of $z = f(x, y)$ at $(0, 0)$.

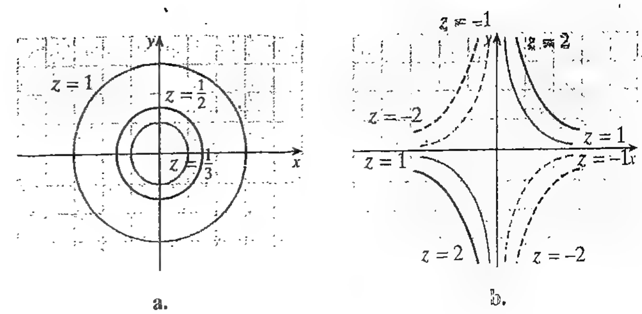


Figure 11.45 Problem 59

60. Prove the second partials test. *Hint*: Compute the second directional derivative of f in the direction of the unit vector $\mathbf{u} = h\mathbf{i} + k\mathbf{j}$ and complete the square.

61. Verify the formulas for m and b associated with the least squares approximation.

62. This problem involves a generalization of the least squares procedure, in which a "least squares plane" is found to produce the best fit for a given set of data. A researcher knows that the quantity z is related to x and y by a formula of the form $z = k_1x + k_2y$, where k_1 and k_2 are physical constants. To determine these constants, she conducts a series of experiments, the results of which are tabulated as follows:

x	1.20	0.86	1.03	1.65	−0.95	−1.07
y	0.43	1.92	1.52	−1.03	1.22	+0.06
z	3.21	5.73	2.22	0.92	−1.11	−0.97

Modify the method of least squares to find a "best approximation" for k_1 and k_2.

11.8 Lagrange Multipliers

IN THIS SECTION method of Lagrange multipliers, constrained optimization problems, Lagrange multipliers with two parameters, a geometric interpretation of Lagrange's theorem

METHOD OF LAGRANGE MULTIPLIERS

In many applied problems, a function of two variables is to be optimized subject to a restriction or **constraint** on the variables. For example, consider a container heated in such a way that the temperature at the point (x, y, z) in the container is given by the function $T(x, y, z)$. Suppose that the surface $z = f(x, y)$ lies in the container, and that we wish to find the point on $z = f(x, y)$ where the temperature is the greatest. In other words, *What is the maximum value of T subject to the constraint $z = f(x, y)$, and where does this maximum value occur?*

***THEOREM 11.14* Lagrange's theorem**

Assume that f and g have continuous first partial derivatives and that f has an extremum at $P_0(x_0, y_0)$ on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, there is a number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Proof Denote the constraint curve $g(x, y) = c$ by C, and note that C is smooth. We represent this curve by the vector function

$$\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

for all t in an open interval I, including t_0 corresponding to P_0, where $x'(t)$ and $y'(t)$ exist and are continuous. Let $F(t) = f(x(t), y(t))$ for all t in I, and apply the chain rule to obtain

$$F'(t) = f_x(x(t), y(t))\frac{dx}{dt} + f_y(x(t), y(t))\frac{dy}{dt} = \nabla f(x(t), y(t)) \cdot \mathbf{R}'(t)$$

Because $f(x, y)$ has an extremum at P_0, we know that $F(t)$ has an extremum at t_0, the value of t that corresponds to P_0 (that is, P_0 is the point on C where $t = t_0$). Therefore, we have $F'(t_0) = 0$ and

$$F'(t_0) = \nabla f(x(t_0), y(t_0)) \cdot \mathbf{R}'(t_0) = 0$$

If $\nabla f(x(t_0), y(t_0)) = 0$, then $\lambda = 0$, and the condition $\nabla f = \lambda \nabla g$ is satisfied trivially. If $\nabla f(x(t_0), y(t_0)) \neq 0$, then $\nabla f(x(t_0), y(t_0))$ is orthogonal to $\mathbf{R}'(t_0)$. Because $\mathbf{R}'(t_0)$ is tangent to the constraint curve C, it follows that $\nabla f(x_0, y_0)$ is normal to C. But $\nabla g(x_0, y_0)$ is also normal to C (because C is a level curve of g), and we conclude that ∇f and ∇g must be *parallel* at P_0. Thus, there is a scalar λ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$, as required. ☐

CONSTRAINED OPTIMIZATION PROBLEMS

The general procedure for the method of Lagrange multipliers may be described as follows.

Procedure for the Method of Lagrange Multipliers

Suppose f and g satisfy the hypotheses of Lagrange's theorem, and that $f(x, y)$ has an extremum subject to the constraint $g(x, y) = c$. Then to find the extreme value, proceed as follows:

1. Simultaneously solve the following three equations for x, y, and λ:

$$f_x(x, y) = \lambda g_x(x, y) \qquad f_y(x, y) = \lambda g_y(x, y) \qquad g(x, y) = c$$

2. Evaluate f at all points found in step 1 and all points on the boundary of the constraint. The extremum we seek must be among these values.

EXAMPLE 1 Optimization with Lagrange multipliers

Given that the largest and smallest values of $f(x, y) = 1 - x^2 - y^2$ subject to the constraint $x + y = 1$ with $x \geq 0$, $y \geq 0$ exist, use the method of Lagrange multipliers to find these extrema.

Solution

Because the constraint is $x + y = 1$, let $g(x, y) = x + y$

$$f_x(x, y) = -2x \qquad f_y(x, y) = -2y \qquad g_x(x, y) = 1 \qquad g_y(x, y) = 1$$

Form the system

$$\begin{cases} -2x = \lambda(1) & \leftarrow f_x(x, y) = \lambda g_x(x, y) \\ -2y = \lambda(1) & \leftarrow f_y(x, y) = \lambda g_y(x, y) \\ x + y = 1 & \leftarrow g(x, y) = 1 \end{cases}$$

SMH This system is solved as Example 3.4 in Section 3.1 of the *Student Mathematics Handbook*.

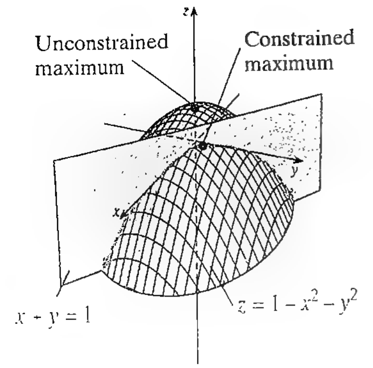


Figure 11.46 The maximum is the high point of the curve of intersection of the surface and the plane

The only solution is $x = \frac{1}{2}$, $y = \frac{1}{2}$.

$$f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 1 - \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2}$$

The endpoints of the line segment

$$x + y = 1 \qquad \text{for } x \geq 0, y \geq 0$$

are at (1, 0) and (0, 1), and we find that

$$f(1, 0) = 1 - 1^2 - 0^2 = 0$$
$$f(0, 1) = 1 - 0^2 - 1^2 = 0$$

Therefore, the maximum value is $\frac{1}{2}$ at $\left(\frac{1}{2}, \frac{1}{2}\right)$, and the minimum value is 0 at (1, 0) and (0, 1). (See Figure 11.46.) ■

The method of Lagrange multipliers extends naturally to functions of three or more variables. If a function $f(x, y, z)$ has an extreme value subject to a constraint $g(x, y, z) = c$, then the extremum occurs at a point (x_0, y_0, z_0) such that $g(x_0, y_0, z_0) = c$ and $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$ for some number λ. Here is an example.

EXAMPLE 2 Hottest and coldest points on a plate

A container in $\mathbb{R}^3$ has the shape of the cube given by $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$. A plate is placed in the container in such a way that it occupies that portion of the plane $x + y + z = 1$ that lies in the cubical container. If the container is heated so that the temperature at each point (x, y, z) is given by

$$T(x, y, z) = 4 - 2x^2 - y^2 - z^2$$

in hundreds of degrees Celsius, what are the hottest and coldest points on the plate? You may assume these extreme temperatures exist.

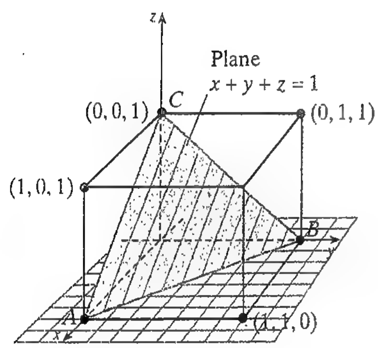


Figure 11.47 Find the hottest and coldest points on the plate inside the cube

Solution

The cube and plate are shown in Figure 11.47. We will use Lagrange multipliers to find all critical points in the interior of the plate, and then we will examine the plate's boundary. To apply the method of Lagrange multipliers, we must solve $\nabla T = \lambda \nabla g$, where $g(x, y, z) = x + y + z$. We obtain the partial derivatives

$$T_x = -4x \qquad T_y = -2y \qquad T_z = -2z \qquad g_x = g_y = g_z = 1$$

We must solve the system

SMH See Problem 29 of Problem Set 3 of the *Student Mathematics Handbook.*

$$\begin{cases} -4x = \lambda & \leftarrow T_x = \lambda g_x \\ -2y = \lambda & \leftarrow T_y = \lambda g_y \\ -2z = \lambda & \leftarrow T_z = \lambda g_z \\ x + y + z = 1 & \leftarrow g(x, y, z) = 1 \end{cases}$$

The solution of this system is $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$. The boundary of the plate is a triangle with vertices $A(1, 0, 0)$, $B(0, 1, 0)$, and $C(0, 0, 1)$. The temperature along the edges of this triangle may be found as follows:

Edge AC: $x + z = 1, y = 0$
Edge AB: $x + y = 1, z = 0$
Edge BC: $y + z = 1, x = 0$

$$T_1(x) = 4 - 2x^2 - (0)^2 - (1 - x)^2 = 3 - 3x^2 + 2x, \quad 0 \leq x \leq 1$$
$$T_2(x) = 4 - 2x^2 - (1 - x)^2 - (0)^2 = 3 - 3x^2 + 2x, \quad 0 \leq x \leq 1$$
$$T_3(y) = 4 - 2(0)^2 - y^2 - (1 - y)^2 = 3 + 2y - 2y^2, \quad 0 \leq y \leq 1$$

Edge AC: Differentiating, $T_1'(x) = T_2'(x) = -6x + 2$, which equals 0 when $x = \frac{1}{3}$. If $x = \frac{1}{3}$, then $z = \frac{2}{3}$ (because $x + z = 1, y = 0$ on edge AC), so we have the critical point $\left(\frac{1}{3}, 0, \frac{2}{3}\right)$.

Edge AB: Because $T_2 = T_1$, we see $x = \frac{1}{3}$. If $x = \frac{1}{3}$, then $y = \frac{2}{3}$ (because $x + y = 1$, $z = 0$ on edge BC), so we have another critical point $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$.

Edge BC: Differentiating, $T_3'(y) = 2-4y$, which equals 0 when $y = \frac{1}{2}$. Because $y + z = 1$ and $x = 0$, we have the critical point $\left(0, \frac{1}{2}, \frac{1}{2}\right)$.

Endpoints of the edges: $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

The last step is to evaluate T at the critical points and the endpoints:

$$T\left(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}\right) = 3\tfrac{3}{5};$$

$$T\left(\tfrac{1}{3}, 0, \tfrac{2}{3}\right) = 3\tfrac{1}{3}; \quad T\left(\tfrac{1}{3}, \tfrac{2}{3}, 0\right) = 3\tfrac{1}{3}; \quad T\left(0, \tfrac{1}{2}, \tfrac{1}{2}\right) = 3\tfrac{1}{2};$$

$$T(1, 0, 0) = 2; \quad T(0, 1, 0) = 3; \quad T(0, 0, 1) = 3$$

Comparing these values (remember that the temperature is in hundreds of degrees Celsius), we see that the highest temperature is 360°C at $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ and the lowest temperature is 200°C at $(1, 0, 0)$. ■

Notice that the multiplier is used only as an intermediary device for finding the critical points and plays no role in the final determination of the constrained extrema. However, the value of λ is more important in certain problems, thanks to the interpretation given in the following theorem.

THEOREM 11.15 Rate of change of the extreme value

Suppose E is an extreme value (maximum or minimum) of f subject to the constraint $g(x, y) = c$. Then the Lagrange multiplier λ is the rate of change of E with respect to c; that is, $\lambda = dE/dc$.

Proof Note that at the extreme value (x, y) we have

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad \text{and} \quad g(x, y) = c$$

The coordinates of the optimal ordered pair (x, y) depend on c (because different constraint levels will generally lead to different optimal combinations of x and y). Thus,

$$E = E(x, y) \quad \text{where } x \text{ and } y \text{ are functions of } c$$

By the chain rule for partial derivatives:

$$\begin{aligned}
\frac{dE}{dc} &= \frac{\partial E}{\partial x}\frac{dx}{dc} + \frac{\partial E}{\partial y}\frac{dy}{dc} \\
&= f_x\frac{dx}{dc} + f_y\frac{dy}{dc} && \text{Because } E = f(x, y) \\
&= \lambda g_x\frac{dx}{dc} + \lambda g_y\frac{dy}{dc} && \text{Because } f_x = \lambda g_x \text{ and } f_y = \lambda g_y \\
&= \lambda\left(g_x\frac{dx}{dc} + g_y\frac{dy}{dc}\right) \\
&= \lambda\frac{dg}{dc} && \text{Chain rule} \\
&= \lambda && \text{Because } \frac{dg}{dc} = 1 \text{ (remember } g = c)
\end{aligned}$$

□

This theorem can be interpreted as saying that the multiplier estimates the change in the extreme value E that results when the constraint c is increased by 1 unit. This interpretation is illustrated in the following example.

EXAMPLE 3 Maximum output for a Cobb-Douglas production function

If x thousand dollars is spent on labor, and y thousand dollars is spent on equipment, it is estimated that the output of a certain factory will be

$$Q(x, y) = 50x^{2/5}y^{3/5}$$

units. If \$150,000 is available, how should this capital be allocated between labor and equipment to generate the largest possible output? How does the maximum output change if the money available for labor and equipment is increased by \$1,000? In economics, an output function of the general form $Q(x, y) = x^{\alpha} y^{1-\alpha}$ is known as a *Cobb-Douglas production function.*

Solution

Because x and y are given in units of \$1,000, the constraint equation is $x + y = 150$. If we set $g(x, y) = x + y$, we wish to maximize Q subject to $g(x, y) = 150$. To apply the method of Lagrange multipliers, we first find

$$Q_x = 20x^{-3/5}y^{3/5} \qquad Q_y = 30x^{2/5}y^{-2/5} \qquad g_x = 1 \qquad g_y = 1$$

Next, solve the system

$$\begin{cases} 20x^{-3/5}y^{3/5} = \lambda(1) \\ 30x^{2/5}y^{-2/5} = \lambda(1) \\ x + y = 150 \end{cases}$$

From the first two equations we have

$$\begin{aligned} 20x^{-3/5}y^{3/5} &= 30x^{2/5}y^{-2/5} \\ 20y &= 30x \\ y &= 1.5x \end{aligned}$$

Substitute $y = 1.5x$ into the equation $x + y = 150$ to find $x = 60$. This leads to the solution $y = 90$, so that the maximum output is

$$Q(60, 90) = 50(60)^{2/5}(90)^{3/5} \approx 3{,}826.273502 \quad \text{units}$$

We also find that

$$\lambda = 20(60)^{-3/5}(90)^{3/5} \approx 25.50849001$$

Thus, the maximum output is about 3,826 units and occurs when \$60,000 is allocated to labor and \$90,000 to equipment. We also note that an increase of \$1,000 (1 unit) in the available funds will increase the maximum output by approximately $\lambda \approx 25.51$ units (from 3,826.27 to 3,851.78 units). ■

LAGRANGE MULTIPLIERS WITH TWO PARAMETERS

The method of Lagrange multipliers can also be applied in situations with more than one constraint equation. Suppose we wish to locate an extremum of a function defined by $f(x, y, z)$ subject to two constraints, $g(x, y, z) = c_1$ and $h(x, y, z) = c_2$, where g and h are also differentiable and ∇g and ∇h are not parallel. By generalizing Lagrange's theorem, it can be shown that if (x_0, y_0, z_0) is the desired extremum, then there are numbers λ and μ such that $g(x_0, y_0, z_0) = c_1$, $h(x_0, y_0, z_0) = c_2$, and

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

As in the case of one constraint, we proceed by first solving this system of equations simultaneously to find λ, μ, x_0, y_0, z_0 and then evaluating $f(x, y, z)$ at each solution and comparing to find the required extremum. This approach is illustrated in our final example of this section.

EXAMPLE 4 Optimization with two constraints

Find the point on the intersection of the plane $x + 2y + z = 10$ and the paraboloid $z = x^2 + y^2$ that is closest to the origin (see Figure 11.48). You may assume that such a point exists.

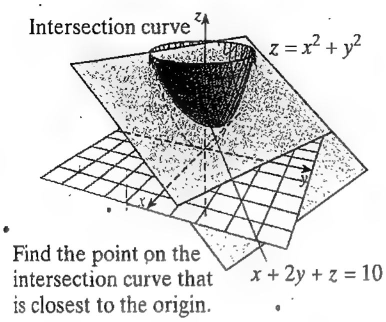


Figure 11.48 Graphical representation of Example 4

Solution

The distance from a point (x, y, z) to the origin is $s = \sqrt{x^2 + y^2 + z^2}$, but instead of minimizing this quantity, it is easier to minimize its square. That is, we will minimize

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the joint constraints

$$g(x, y, z) = x + 2y + z = 10 \quad \text{and} \quad h(x, y, z) = x^2 + y^2 - z = 0$$

Compute the partial derivatives of f, g, and h:

$$\begin{array}{lll} f_x = 2x & f_y = 2y & f_z = 2z \\ g_x = 1 & g_y = 2 & g_z = 1 \\ h_x = 2x & h_y = 2y & h_z = -1 \end{array}$$

To apply the method of Lagrange multipliers, we use the formula

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

which leads to the following system of equations:

$$\begin{cases} 2x = \lambda(1) + \mu(2x) \\ 2y = \lambda(2) + \mu(2y) \\ 2z = \lambda(1) + \mu(-1) \\ x + 2y + z = 10 \\ z = x^2 + y^2 \end{cases}$$

This is not a linear system, so solving it requires ingenuity.

Multiply the first equation by 2 and subtract the second equation to obtain

$$\begin{aligned} 4x - 2y &= (4x - 2y)\mu \\ (4x - 2y) - (4x - 2y)\mu &= 0 \\ (4x - 2y)(1 - \mu) &= 0 \\ 4x - 2y = 0 \quad &\text{or} \quad 1 - \mu = 0 \end{aligned}$$

CASE I: **If $4x - 2y = 0$**, then $y = 2x$. Substitute this into the two constraint equations:

$x + 2y + z = 10$	$x^2 + y^2 - z = 0$
$x + 2(2x) + z = 10$	$x^2 + (2x)^2 - z = 0$
$z = 10 - 5x$	$z = 5x^2$

By substitution we have $5x^2 = 10 - 5x$, which has solutions $x = 1$ and $x = -2$. This implies

$x = 1$	$x = -2$
$y = 2x = 2(1) = 2$	$y = 2x = 2(-2) = -4$
$z = 5x^2 = 5(1)^2 = 5$	$z = 5x^2 = 5(-2)^2 = 20$

Thus, the points $(1, 2, 5)$ and $(-2, -4, 20)$ are candidates for the minimal distance.

CASE II: **If $1 - \mu = 0$**, then $\mu = 1$, and we look at the system of equations involving x, y, z, λ, and μ.

$$\begin{cases} 2x = \lambda(1) + \mu(2x) \\ 2y = \lambda(2) + \mu(2y) \\ 2z = \lambda(1) + \mu(-1) \\ x + 2y + z = 10 \\ z = x^2 + y^2 \end{cases}$$

The top equation becomes $2x = \lambda + 2x$, so that $\lambda = 0$. We now find z from the third equation:

$$2z = -1 \quad \text{or} \quad z = -\tfrac{1}{2}$$

Next, turn to the constraint equations:

$$\begin{array}{c|c} x + 2y + z = 10 & x^2 + y^2 - z = 0 \\ \hline x + 2y - \frac{1}{2} = 10 & x^2 + y^2 + \frac{1}{2} = 0 \\ x + 2y = 10 + \frac{1}{2} & x^2 + y^2 = -\frac{1}{2} \end{array}$$

There is no solution because $x^2 + y^2$ cannot equal a negative number.

We check the candidates for the minimal distance:

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 + z^2 \quad \text{so that} \\ f(1, 2, 5) &= 1^2 + 2^2 + 5^2 = 30 \\ f(-2, -4, 20) &= (-2)^2 + (-4)^2 + 20^2 = 420 \end{aligned}$$

Because $f(x, y, z)$ represents the square of the distance, the minimal distance is $\sqrt{30}$ and the point on the intersection of the two surfaces nearest to the origin is $(1, 2, 5)$. ■

A GEOMETRIC INTERPRETATION

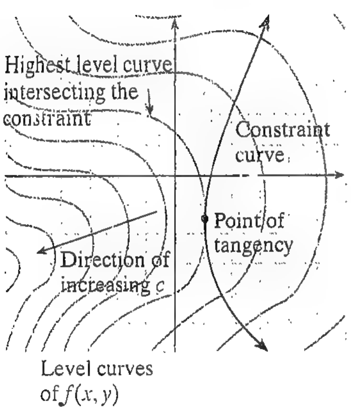


Figure 11.49 Increasing level curves and the constraint curve

Lagrange's theorem can be interpreted geometrically. Suppose the constraint curve $g(x, y) = c$ and the level curves $f(x, y) = k$ are drawn in the xy-plane, as shown in Figure 11.49.

To maximize $f(x, y)$ subject to the constraint $g(x, y) = c$, we must find the "highest" (leftmost, actually) level curve of f that intersects the constraint curve. As Figure 11.49 suggests, this critical intersection occurs at a point where the constraint curve is tangent to a level curve—that is, where the slope of the constraint curve $g(x, y) = c$ is equal to the slope of a level curve $f(x, y) = k$. According to the formula derived in Section 11.5 (p. 732),

$$\text{Slope of constraint curve } g(x, y) = c \text{ is } \frac{-g_x}{g_y}$$

$$\text{Slope of each level curve is } \frac{-f_x}{f_y}.$$

The condition that the slopes are equal can be expressed by

$$\frac{-f_x}{f_y} = \frac{-g_x}{g_y}, \qquad \text{or, equivalently,} \qquad \frac{f_x}{g_x} = \frac{f_y}{g_y}$$

Let λ equal this common ratio,

$$\lambda = \frac{f_x}{g_x} \qquad \text{and} \qquad \lambda = \frac{f_y}{g_y}$$

so that

$$f_x = \lambda g_x \qquad \text{and} \qquad f_y = \lambda g_y$$

and

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} = \lambda(g_x\mathbf{i} + g_y\mathbf{j}) = \lambda \nabla g$$

Because the point in question must lie on the constraint curve, we also have $g(x, y) = c$. If these equations are satisfied at a certain point (a, b), then f will reach its constrained *maximum* at (a, b) if the *highest* level curve that intersects the constraint curve does so at this point. On the other hand, if the *lowest* level curve that intersects the constraint curve does so at (a, b), then f achieves its constrained *minimum* at this point.

11.8 PROBLEM SET

Guidelines §1–31, 41–44, 51–54.

In the problems in this set, you may assume that the requested extreme value(s) exist.

A *Use the method of Lagrange multipliers to find the required constrained extrema in Problems 1–14.*

1. Maximize $f(x, y) = xy$ subject to $2x + 2y = 5$.
2. Maximize $f(x, y) = xy$ subject to $x + y = 20$.
3. Maximize $f(x, y) = 16 - x^2 - y^2$ subject to $x + 2y = 6$.
4. Minimize $f(x, y) = x^2 + y^2$ subject to $x + y = 24$.
5. Minimize $f(x, y) = x^2 + y^2$ subject to $xy = 1$.
6. Minimize $f(x, y) = x^2 - xy + 2y^2$ subject to $2x + y = 22$.
7. Minimize $f(x, y) = x^2 - y^2$ subject to $x^2 + y^2 = 4$.
8. Maximize $f(x, y) = x^2 - 2y - y^2$ subject to $x^2 + y^2 = 1$.
9. Maximize $f(x, y) = \cos x + \cos y$ subject to $y = x + \frac{\pi}{4}$.
10. Maximize $f(x, y) = e^{xy}$ subject to $x^2 + y^2 = 3$.
11. Maximize $f(x, y) = \ln(xy^2)$ subject to $2x^2 + 3y^2 = 8$ for $x > 0$.
12. Maximize $f(x, y, z) = xyz$ subject to $3x + 2y + z = 6$.
13. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x - 2y + 3z = 4$.
14. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to $4x^2 + 2y^2 + z^2 = 4$.

B 15. Find the smallest value of $f(x, y, z) = 2x^2 + 4y^2 + z^2$ subject to $4x - 8y + 2z = 10$. What, if anything, can be said about the largest value of f subject to this constraint?

16. Let $f(x, y, z) = x^2y^2z^2$. Show that the maximum value of f on the sphere $x^2 + y^2 + z^2 = R^2$ is $R^6/27$.

17. Find the maximum and minimum values of $f(x, y, z) = x - y + z$ on the sphere $x^2 + y^2 + z^2 = 100$.

18. Find the maximum and minimum values of $f(x, y, z) = 4x - 2y - 3z$ on the sphere $x^2 + y^2 + z^2 = 100$.

19. Use Lagrange multipliers to find the distance from the origin to the plane $Ax + By + Cz = D$ where at least one of A, B, C is nonzero.

20. Find the maximum and minimum distance from the origin to the ellipse $5x^2 - 6xy + 5y^2 = 4$.

21. Find the point on the plane $2x + y + z = 1$ that is nearest to the origin.

BCW 22

22. Find the largest product of positive numbers x, y, and z such that their sum is 24.

23. Write the number 12 as the sum of three positive numbers x, y, z in such a way that the product xy^2z is a maximum.

BCW 23

24. A rectangular box with no top is to be constructed from 96 ft^2 of material. What should be the dimensions of the box if it is to enclose maximum volume?

25. The temperature T at point (x, y, z) in a region of space is given by the formula $T = 100 - xy - xz - yz$. Find the lowest temperature on the plane $x + y + z = 10$.

26. A farmer wishes to fence off a rectangular pasture along the bank of a river. The area of the pasture is to be 3,200 yd^2, and no fencing is needed along the river bank. Find the dimensions of the pasture that will require the least amount of fencing.

27. There are 320 yd of fencing available to enclose a rectangular field. How should the fencing be used so that the enclosed area is as large as possible?

may be left

28. Use the fact that 12 oz is approximately 6.89π in.^{3} to find the dimensions of the 12-oz Coke can that can be constructed using the least amount of metal. Compare your answer with an actual can of Coke. Explain what might cause the discrepancy.

29. A cylindrical can is to hold 4π in.^{3} of orange juice. The cost per square inch of constructing the metal top and bottom is twice the cost per square inch of constructing the cardboard side. What are the dimensions of the least expensive can?

30. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

31. A manufacturer has \$8,000 to spend on the development and promotion of a new product. It is estimated that if x thousand dollars is spent on development and y thousand is spent on promotion, sales will be approximately $f(x, y) = 50x^{1/2}y^{3/2}$ units. How much money should the manufacturer allocate to development and how much to promotion to maximize sales?

32. **Modeling Problem** If x thousand dollars is spent on labor and y thousand dollars is spent on equipment, the output at a certain factory may by modeled by

$$Q(x, y) = 60x^{1/3}y^{2/3}$$

units. Assume \$120,000 is available.

a. How should money be allocated between labor and equipment to generate the largest possible output?

b. Use the Lagrange multiplier λ to estimate the change in the maximum output of the factory that would result if the money available for labor and equipment is increased by \$1,000.

33. **Modeling Problem** An architect decides to model the usable living space in a building by the volume of space that can be used comfortably by a person 6 feet tall—that is, by the largest 6-foot-high rectangular box that can be inscribed in the building. Find the dimensions of an A-frame building y ft long with equilateral triangular ends x ft on a side that maximizes usable living space if the exterior surface area of the building cannot exceed 500 ft^2.

34. Find the radius of the largest cylinder of height 6 in. that can be inscribed in an inverted cone of height H, radius R, and lateral surface area 250 in.^{2}.

35. In Problem 42 of Problem Set 11.7, you were asked to minimize the ground state energy

$$E(x, y, z) = \frac{k^2}{8m}\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$$

subject to the volume constraint $V = xyz = C$. Solve the problem using Lagrange multipliers.

36. **Modeling Problem** A university extension agricultural service concludes that, on a particular farm, the yield of wheat per acre is a function of water and fertilizer. Let x be the number of acre-feet of water applied, and y the number of pounds of fertilizer applied during the growing season. The agricultural

service then concludes that the yield B (measured in bushels), can be modeled by the formula $B(x, y) = 500 + x^2 + 2y^2$. Suppose that water costs \$20 per acre-foot, fertilizer costs \$12 per pound, and the farmer will invest \$236 per acre for water and fertilizer. How much water and fertilizer should the farmer buy to maximize the yield?

37. How would the farmer of Problem 36 maximize the yield if the amount spent is \$100 instead of \$236?

38. Present post office regulations specify that a box (that is, a package in the form of a rectangular parallelepiped) can be mailed parcel post only if the sum of its length and girth does not exceed 108 inches, as shown in Figure 11.50. Find the maximum volume of such a package. (Compare your solution here with the one you might have given to Problem 14, Section 4.6)

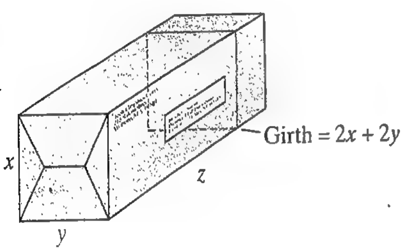


Figure 11.50 Maximum volume for a box mailed by U.S. parcel post

39. *Heron's formula* says that the area of a triangle with sides a, b, c is

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$ is the semiperimeter of the triangle. Use this result and the method of Lagrange multipliers to show that of all triangles with a given fixed perimeter P, the equilateral triangle has the largest area.

40. If x, y, z are the angles of a triangle, what is the maximum value of the product $P(x, y, z) = \sin x \sin y \sin z$? What about $Q(x, y, z) = \cos x \cos y \cos z$?

Use the method of Lagrange multipliers in Problems 41–44 to find the required extrema for the two given constraints.

41. Find the minimum of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x + y = 4$ and $y + z = 6$.

42. Find the maximum of $f(x, y, z) = xyz$ subject to $x^2 + y^2 = 3$ and $y = 2z$.

43. Maximize $f(x, y, z) = xy + xz$ subject to $2x + 3z = 5$ and $xy = 4$.

44. Minimize $f(x, y, z) = 2x^2 + 3y^2 + 4z^2$ subject to $x + y + z = 4$ and $x - 2y + 5z = 3$.

45. **Modeling Problem** A manufacturer is planning to sell a new product at the price of \$150 per unit and estimates that if x thousand dollars is spent on development and y thousand dollars on promotion, then approximately

$$\frac{320y}{y+2} + \frac{160x}{x+4}$$

units of the product will be sold. The cost of manufacturing the product is \$50 per unit.

a. If the manufacturer has a total of \$8,000 to spend on the development and promotion, how should this money be allocated to generate the largest possible profit?

b. Suppose the manufacturer decides to spend \$8,100 instead of \$8,000 on the development and promotion of the new product. Estimate how this change will affect the maximum possible profit.

c. If unlimited funds are available, how much should the manufacturer spend on development and promotion to maximize profit?

d. What is the Lagrange multiplier in part c? Your answer should suggest another method for solving the problem in part c. Solve the problem using this alternative approach.

46. **Modeling Problem** A jewelry box with a square base has an interior partition and is required to have volume 800 cm³ (see Figure 11.51).

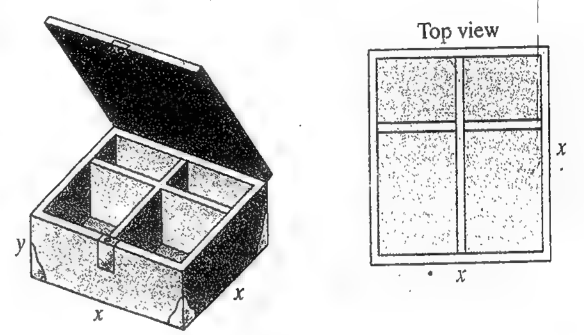


Figure 11.51 Constructing a jewelry box

a. The material in the top costs twice as much as that in the sides and bottom, which in turn, costs twice as much as the material in the partitions. Find the dimensions of the box that minimize the total cost of construction. *Does it matter that you have not been told where the partitions are located?*

b. Suppose the volume constraint changes from 800 cm³ to 801 cm³. Estimate the appropriate effect on the minimal cost.

47. Three alleles, A, B, and O, determine the four blood types A, B, O, and AB. The *Hardy-Weinberg law* says that the proportions of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where p, q, and r are the proportions of blood types A, B, and O in the population. Given that $p + q + r = 1$, what is the largest value of P?

48. A farmer wants to build a metal silo in the shape of a right circular cylinder with a right circular cone on the top (the bottom of the silo will be a concrete slab). What is the least amount of metal that can be used if the silo is to have a fixed volume V_0?

49. Find the volume of the largest rectangular parallelepiped (box) that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(See Problem 30.)

50. Exploration Problem The method of Lagrange multipliers gives a constrained extremum only if one exists. If the method is applied to optimizing $f(x, y) = x + y$ subject to $xy = 1$, the method yields two candidates for an extremum. Is one a maximum and the other a minimum? Explain.

In Problems 51–54, let $Q = f(x, y)$ be a production function in which x and y represent units of labor and capital, respectively. If p and q represent unit costs of labor and capital, respectively, then $C(x, y) = px + qy$ represents the total cost of production.

51. Use Lagrange multipliers to show that subject to a fixed production level Q_0, the total cost is smallest when

$$\frac{f_x}{p} = \frac{f_y}{q} \quad \text{and} \quad Q = f(x, y) = Q_0$$

(provided $\nabla f \neq \mathbf{0}$, and $p \neq 0$, $q \neq 0$). This is often referred to as the *minimum cost problem*, and its solution is called the *least cost combination of inputs*.

52. Show that the inputs x, y that maximize the production level $Q = f(x, y)$ subject to a fixed cost k satisfy

$$\frac{f_x}{p} = \frac{f_y}{q} \quad \text{with } px + qy = k$$

(assume $p \neq 0$, $q \neq 0$). This is called a *fixed-budget problem*.

53. A *Cobb-Douglas production function* is an output function of the form $Q(x, y) = cx^{\alpha} y^{\beta}$, with $\alpha + \beta = 1$.

a. Show that such a function is maximized with respect to the fixed cost $px + qy = k$ when $x = \alpha k/p$ and $y = \beta k/q$.

b. Where does the maximum occur if we drop the condition $\alpha + \beta = 1$? How does the maximum output change if k is increased by 1 unit?

54. Show that the cost function

$$C(x, y) = px + qy$$

is minimized subject to the fixed production level $Ax^{\alpha}y^{\beta} = k$ with $\alpha + \beta = 1$ when

$$x = \frac{k}{A}\left(\frac{\alpha q}{\beta p}\right)^{\beta} \qquad y = \frac{k}{A}\left(\frac{\beta p}{\alpha q}\right)^{\alpha}$$

55. HISTORICAL QUEST Joseph Lagrange is generally acknowledged as one of the two greatest mathematicians of the eighteenth century, along with Leonhard Euler (see **HISTORICAL QUEST #58** of the Supplementary Problems of Chapter 4). There is a distinct difference in style between Lagrange and Euler. Lagrange has been characterized as the first true analyst in the sense that he attempted to write concisely and with rigor. On the other hand, Euler wrote using intuition and with an abundance of detail. Lagrange was described by Napoleon Bonaparte as "the lofty pyramid of the mathematical sciences," and followed Euler as the court mathematician for Frederick the Great. He was the first to use the notation $f'(x)$ and $f''(x)$ for derivatives. In this section we were introduced to the method of Lagrange multipliers, which provide a procedure for constrained optimization. This method was contained in a paper on mechanics that Lagrange wrote when he was only 19 years old.* ■


JOSEPH LAGRANGE
1736–1813

For this Quest, we consider Lagrange's work with solving *algebraic* equations. You are familiar with the quadratic formula, which provides a general solution for any second-degree equation $ax^2 + bx + c = 0$, $a \neq 0$. Lagrange made an exhaustive study of the general solution for the first four degrees. Here is what he did. Suppose you are given a general algebraic expression involving letters $a, b, c, \ldots$; how many *different* expressions can be derived from the given one if the letters are interchanged in all possible ways? For example, from $ab + cd$ we obtain $ad + cb$ by interchanging b and d.

This problem suggests another closely related problem, also part of Lagrange's approach. Lagrange solved general algebraic equations of degrees 2, 3, and 4. It was proved later (not by Lagrange, but by Galois and Abel) that no general solution for equations of degree greater than 4 can be found. Do some research and find the general solution for equations of degrees 1, 2, 3, and 4.

*From *Men of Mathematics* by E. T. Bell, Simon & Schuster, New York, 1937, p. 165.

CHAPTER 11 REVIEW

Proficiency Examination

CONCEPT PROBLEMS

1. What is a function of two variables?
2. What are the domain and range of a function of two variables?
3. What is a level curve of the function defined by $f(x, y)$?
4. What do we mean by the limit of a function of two variables?
5. State the following properties of a limit of functions of two variables.
   a. scalar rule **b.** sum rule
   c. product rule **d.** quotient rule
6. Define the continuity of a function defined by $f(x, y)$ at a point (x_0, y_0) in its domain and continuity on a set S.
7. If $z = f(x, y)$, define the first partial derivatives of f with respect to x and y.

8. What is the slope of a tangent line to the surface defined by $z = f(x, y)$ that is parallel to the xy-plane at a point P_0 on f?
9. If $z = f(x, y)$, represent the second partial derivatives.
10. If $z = f(x, y)$, what are the increments of x, y, and z?
11. What does it mean for a function of two variables to be differentiable at (x_0, y_0)?
12. State the incremental approximation of $f(x, y)$.
13. Define the total differential of $z = f(x, y)$.
14. State the chain rule for a function of one parameter.
15. State the chain rule for a function of two independent parameters.
16. Define the directional derivative of a function defined by $z = f(x, y)$.
17. Define the gradient $\nabla f(x, y)$.
18. State the following basic properties of the gradient.

    a. constant rule　　b. linearity rule
    c. product rule　　d. quotient rule
    e. power rule
19. Express the directional derivative in terms of the gradient.
20. State the optimal direction property of the gradient (that is, the steepest ascent and steepest descent).
21. State the normal property of the gradient.
22. Define the normal line and tangent plane to a surface S at a point P_0.
23. Define the absolute extrema of a function of two variables.
24. Define the relative extrema of a function of two variables.
25. What is a critical point of a function of two variables?
26. State the second partials test.
27. State the extreme value theorem for a function of two variables.
28. What is the least squares approximation of data, and what is a regression line?
29. State Lagrange's theorem.
30. State the procedure for the method of Lagrange multipliers.

PRACTICE PROBLEMS

31. If $f(x, y) = \sin^{-1} xy$, verify that $f_{xy} = f_{yx}$.
32. Let $w = x^2y + y^2z$, where $x = t \sin t$, $y = t \cos t$, and $z = 2t$. Use the chain rule to find $\frac{dw}{dt}$, where $t = \pi$.
33. Let $f(x, y, z) = xy + yz + xz$, and let P_0 denote the point $(1, 2, -1)$.

    a. Find the gradient of f at P_0.
    b. Find the directional derivative of f in the direction from P_0 toward the point $Q(-1, 1, -1)$.
    c. Find the direction from P_0 in which the directional derivative has its largest value. What is the magnitude of the largest directional derivative at P_0?
34. Show that the function defined by
$$f(x, y) = \begin{cases} \dfrac{x^2y}{x^3 + y^3} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$
is not continuous at $(0, 0)$.
35. If $f(x, y) = \ln\left(\frac{y}{x}\right)$, find f_x, f_y, f_{yy}, and f_{xy}.
36. Show that if $f(x, y, z) = x^2y + y^2z + z^2x$, then
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = (x + y + z)^2$$
37. Let $f(x, y) = (x^2 + y^2)^2$. Find the directional derivative of f at $(2, -2)$ in the direction that makes an angle of $\frac{2\pi}{3}$ with the positive x-axis.
38. Find all critical points of $f(x, y) = 12xy - 2x^2 - y^4$ and classify them using the second partials test.
39. Use the method of Lagrange multipliers to find the maximum and minimum values of the function
$$f(x, y) = x^2 + 2y^2 + 2x + 3$$
subject to the constraint $x^2 + y^2 = 4$.
40. Find the largest and smallest values of the function
$$f(x, y) = x^2 - 4y^2 + 3x + 6y$$
on the region defined by $-2 \leq x \leq 2$, $0 \leq y \leq 1$.

Supplementary Problems

Describe the domain of each function given in Problems 1–4.

1. $f(x, y) = \sqrt{16 - x^2 - y^2}$
2. $f(x, y) = \dfrac{x^2 - y^2}{x - y}$
3. $f(x, y) = \sin^{-1} x + \cos^{-1} y$
4. $f(x, y) = e^{x+y} \tan^{-1}\left(\dfrac{y}{x}\right)$

Find the partial derivatives f_x and f_y for the functions defined in Problems 5–10.

5. $f(x, y) = \dfrac{x^2 - y^2}{x + y}$
6. $f(x, y) = x^3 e^{3y/(2x)}$
7. $f(x, y) = x^2y + \sin \dfrac{y}{x}$
8. $f(x, y) = \ln\left(\dfrac{xy}{x + 2y}\right)$
9. $f(x, y) = 2x^3y + 3xy^2 + \dfrac{y}{x}$
10. $f(x, y) = xye^{xy}$

For each function given in Problems 11–15, describe the level curve or level surface $f = c$ for the given values of the constant c.

11. $f(x, y) = x^2 - y$; $c = 2$, $c = -2$
12. $f(x, y) = 6x + 2y$; $c = 0$, $c = 1$, $c = 2$
13. $z = f(x, y) = \begin{cases} \sqrt{x^2 + y^2} & \text{if } x \geq 0 \\ -|y| & \text{if } x < 0 \end{cases}$

    $c = 0$, $c = 1$, $c = -1$
14. $f(x, y, z) = x^2 + y^2 + z^2$; $c = 16$, $c = 0$, $c = -25$
15. $f(x, y, z) = x^2 + \dfrac{y^2}{2} + \dfrac{z^2}{9}$; $c = 1$, $c = 2$

Evaluate the limits in Problems 16 and 17, assuming they exist.

16. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy}{x^2 + y^2}$
17. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x + ye^x}{1 + x^2}$

Show that each limit in Problems 18 and 19 does not exist.

18. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^3 + y^3}$
19. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6 + y^4}$

Find the derivatives in Problems 20–23 using the chain rule. You may leave your answers in terms of x, y, t, u, and v.

20. Find $\frac{dz}{dt}$, where $z = -xy + y^3$, and $x = -3t^2$, $y = 1 + t^3$.

21. Find $\frac{dz}{dt}$, where $z = xy + y^2$, and $x = e^t t^{-1}$, $y = \tan t$.

22. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$, where $z = x^2 - y^2$, and $x = u + 2v$, $y = u - 2v$.

23. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$, where $z = x \tan \frac{x}{y}$, and $x = uv$, $y = \frac{u}{v}$.

Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in Problems 24–27.

24. $x^2 + 6y^2 + 2z^2 = 5$
25. $e^x + e^y + e^z = 3$
26. $x^3 + 2xz - yz^2 - z^3 = 1$
27. $x + 2y - 3z = \ln z$

In Problems 28–33, find f_{xx} and f_{yx}.

28. $f(x, y) = \tan^{-1} xy$
29. $f(x, y) = \sin^{-1} xy$
30. $f(x, y) = x^2 + y^3 - 2xy^2$
31. $f(x, y) = e^{x^2+y^2}$
32. $f(x, y) = x \ln y$
33. $f(x, y) = \int_x^y \sin(\cos t)\, dt$

Find equations for the tangent plane and normal line to the surfaces given in Problems 34–36 at the prescribed point.

34. $x^2y^3z = 8$ at $P_0(2, -1, -2)$
35. $x^3 + 2xy^2 - 7x^3 + 3y + 1 = 0$ at $P_0(1, 1, 1)$
36. $z = \dfrac{-4}{2 + x^2 + y^2}$ at $P_0(1, 1, -1)$

Find all critical points of f(x, y) in Problems 37–42 and classify each as a relative maximum, a relative minimum, or a saddle point.

37. $f(x, y) = x^2 - 6x + 2y^2 + 4y - 2$
38. $f(x, y) = x^3 + y^3 - 6xy$
39. $f(x, y) = (x - 1)(y - 1)(x + y - 1)$
40. $f(x, y) = x^2 + y^3 + 6xy - 7x - 6y$
41. $f(x, y) = x^3 + y^3 + 3x^2 - 18y^2 + 81y + 5$
42. $f(x, y) = \sin(x + y) + \sin x + \sin y$ for $0 < x < \pi$, $0 < y < \pi$ (See Figure 11.52.)

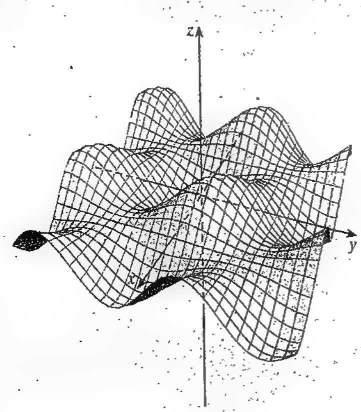


Figure 11.52 Graph of $f(x, y) = \sin(x + y) + \sin x + \sin y$

In Problems 43–46, find the largest and smallest values of the function f on the specified closed, bounded set S.

43. $f(x, y) = xy - 2y$; S is the rectangular region $0 \le x \le 3$, $-1 \le y \le 1$
44. $f(x, y) = x^2 + 2y^2 - x - 2y$; S is the triangular region with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$.
45. $f(x, y) = x^2 + y^2 - 3y$; S is the disk $x^2 + y^2 \le 4$
46. $f(x, y) = 6x - x^2 + 2xy - y^4$; S is the square $0 \le x \le 3$, $0 \le y \le 3$

47. Use the chain rule to find $\frac{dz}{dt}$ if $z = x^2 - 3xy^2$; $x = 2t$, $y = t^2$.
48. Use the chain rule to find $\frac{dz}{dt}$ if $z = x \ln y$; $x = 2t$, $y = e^t$.
49. Let $z = ue^{u^2-v^2}$, where $u = 2x^2 + 3y^2$ and $v = 3x^2 - 2y^2$. Use the chain rule to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
50. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, where x, y, and z are related by the equation $x^3 + 2xz - yz^2 - z^3 = 1$.
51. Find the slope of the level curve of $x^2 + y^2 = 2$, where $x = 1$, $y = 1$.
52. Find the slope of the level curve of $xe^y = 2$, where $x = 2$.
53. Find the equations for the tangent plane and normal line to the surface $z = \sin x + e^{xy} + 2y$ at the point $P_0(0, 1, 3)$.
54. The electric potential at each point (x, y) in the disk $x^2 + y^2 < 4$ is $V = 2(4 - x^2 - y^2)^{-1/2}$ volts. Draw the equipotential curves $V = c$ for $c = \sqrt{2}$, $2/\sqrt{3}$, and 8.
55. Let $f(x, y, z) = x^3y + y^3z + z^3x$. Find a function $g(x, y, z)$ such that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = x^3 + y^3 + z^3 + 3g(x, y, z)$.
56. Let $u = \sin \frac{x}{y} + \ln \frac{y}{x}$. Show that $y\frac{\partial u}{\partial y} + x\frac{\partial u}{\partial x} = 0$.
57. Let $w = \ln(1 + x^2 + y^2) - 2\tan^{-1} y$, where $x = \ln(1 + t^2)$ and $y = e^t$. Use the chain rule to find $\frac{dw}{dt}$.
58. Let $f(x, y) = \tan^{-1} \frac{y}{x}$. Find the directional derivative of f at $(1, 2)$ in the direction that makes an angle of $\frac{\pi}{3}$ with the positive x-axis.
59. Let $f(x, y) = y^x$. Find the directional derivative of f at $P_0(3, 2)$ in the direction toward the point $Q(1, 1)$.
60. According to postal regulations, the largest cylindrical can that can be sent has a girth $(2\pi r)$ plus length ℓ of 108 inches. What is the largest volume cylindrical can that can be mailed?
61. Let $f(x, y, z) = z(x - y)^5 + xy^2z^3$.
    a. Find the directional derivative of f at $(2, 1, -1)$ in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 6$.
    b. In what direction is the directional derivative at $(2, 1, -1)$ largest?
62. Find positive numbers x and y for which xyz is a maximum, given that $x + y + z = 1$. Assume that the extreme value exists.
63. Maximize $f(x, y, z) = x^2yz$ given that x, y, and z are all positive numbers and $x + y + z = 12$. Assume that the extreme value exists.

64. If $z = f(x^2 - y^2)$, evaluate $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y}$.

65. Find the shortest distance from the origin to the surface $z^2 = 3 + xy$. Assume the extreme value exists.

66. **Modeling Problem** A plate is heated in such a way that its temperature at a point (x, y) measured in centimeters on the plate is given in degrees Celsius by

$$T(x, y) = \frac{64}{x^2 + y^2 + 4}$$

a. Find the rate of change in temperature at the point (3, 4) in the direction $2\mathbf{i} + \mathbf{j}$.

b. Find the direction and the magnitude of the greatest rate of change of the temperature at the point (3, 4).

67. **Modeling Problem** The beautiful patterns on the wings of butterflies have long been a subject of curiosity and scientific study. Mathematical models used to study these patterns often focus on determining the level of morphogen (a chemical that affects change). In a model dealing with eyespot patterns, a quantity of morphogen is released from an eyespot and the morphogen concentration t days later is modeled by

$$S(r, t) = \frac{1}{\sqrt{4\pi t}} e^{-(\gamma k t + r^2/(4t))} \qquad t > 0$$

where r measures the radius of the region on the wing affected by the morphogen, and k and γ are positive constants.*

a. Find t_m so that $\partial S/\partial t = 0$. Show that the function $S_m(t)$ formed from $S(r, t)$ by fixing r has a relative maximum at t_m. Is this the same as saying that the function of two variables $S(r, t)$ has a relative maximum?

b. Let $M(r)$ denote the maximum found in part **a**; that is, $M(r) = S(r, t_m)$. Find an expression for M in terms of $z = (1 + 4\gamma\, kr^2)^{1/2}$.

c. Show that $\dfrac{dM}{dr} < 0$ and interpret this result.

68. **Modeling Problem** Certain malignant tumors that do not respond to conventional methods of treatment (surgery, chemotherapy, etc.) may be treated by *hyperthermia*, a process involving the application of extreme heat using microwave transmission (see Figure 11.53). For one particular kind of microwave application used in such therapy, the temperature at each point located r units from the central axis of the tumor and h units inside is modeled by the formula

$$T(r, h) = Ke^{-pr^2}[e^{-qh} - e^{-sh}]$$

where K, p, q, and s are positive constants that depend on the properties of the patient's blood and the heating application.†

a. At what depth inside the tumor does the maximum temperature occur? Express your answers in terms of K, p, q, and s.

b. The article on which this problem is based discusses the physiology of hyperthermia in addition to raising several other interesting mathematical issues. Read this article and discuss assumptions made in the model.

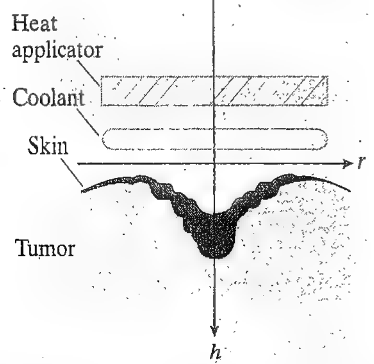


Figure 11.53 Hyperthermia treatment

69. **Modeling Problem** The marketing manager for a certain company has compiled the following data relating monthly advertising expenditure and monthly sales (in units of \$1,000).

Advertising	3	4	7	9	10
Sales	78	86	138	145	156

a. Plot the data on a graph and find the least squares line.

b. Use the least squares line to predict monthly sales if the monthly advertising expenditure is \$5,000.

70. Find $f_{xx} - f_{xy} + f_{yy}$, where $f(x, y) = x^2y^3 + x^3y^2$.

71. Find f_{xyz}, where $f(x, y, z) = \cos(x^2 + y^3 + z^4)$.

72. Let $z = f(x, y)$, where $x = t + \cos t$ and $y = e^t$.

a. Suppose $f_x(1, 1) = 4$ and $f_y(1, 1) = -3$. Find $\dfrac{dz}{dt}$ when $t = 0$.

b. Suppose $f_x(0, 2) = -1$ and $f_y(0, 2) = 3$. Find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ at the point where $r = 2$, $\theta = \frac{\pi}{2}$, and $x = r\cos\theta$, $y = r\sin\theta$.

73. Suppose f has continuous partial derivatives in some region D in the plane, and suppose $f(x, y) = 0$ for all (x, y) in D. If $(1, 2)$ is in D and $f_x(1, 2) = 4$ and $f_y(1, 2) = 6$, find dy/dx when $x = 1$ and $y = 2$.

74. Suppose $\nabla f(x, y, z)$ is parallel to the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ for all (x, y, z). Show that $f(0, 0, a) = f(0, 0, -a)$ for any a.

75. Find two unit vectors that are normal to the surface given by $z = f(x, y)$ at $(0, 1)$, where $f(x, y) = \sin x + e^{xy} + 2y$.

76. Let $f(x, y) = 3(x - 2)^2 - 5(y + 1)^2$. Find all points on the graph of f where the tangent plane is parallel to the plane $2x + 2y - z = 0$.

77. Let z be defined implicitly as a function of x and y by the equation $\cos(x + y) + \cos(x + z) = 1$. Find $\dfrac{\partial^2 z}{\partial y \partial x}$ in terms of x, y, and z.

78. Suppose F and F' are continuous functions of t and that $F'(t) = C$. Define f by $f(x, y) = F(x^2 + y^2)$. Show that the direction of $\nabla f(a, b)$ is the same as the direction of the line joining (a, b) to $(0, 0)$.

79. Let $f(x, y) = 12x^{-1} + 18y^{-1} + xy$, where $x > 0$, $y > 0$. How do you know that f must necessarily have a minimum in the region $x > 0$, $y > 0$? Find the minimum.

*J. D. Murray, *Mathematical Biology*, 2nd edition, Springer-Verlag, New York, 1993, p. 464.

†"Heat Therapy for Tumors," by Leah Edelstein-Keshet, *UMAP Modules 1991: Tools for Teaching*, Consortium for Mathematics and Its Applications, Inc., Lexington, MA, 1992, pp. 73–101.

80. Let $f(x, y) = 3x^4 - 4x^2y + y^2$. Show that f has a minimum at $(0, 0)$ on every line $y = mx$ that passes through the origin. Then show that f has no relative minimum at $(0, 0)$.

In Problems 81–83, you may assume the required extremum exists.

81. Find the minimum of $x^2 + y^2 + z^2$ subject to the constraint $ax + by + cz = 1$ (with $a \neq 0, b \neq 0, c \neq 0$).

82. Suppose $0 < a < 1$ and $x \geq 0, y \geq 0$. Find the maximum of $x^a y^{1-a}$ subject to the constraint $ax + (1 - a)y = 1$.

83. The **geometric mean** of three positive numbers x, y, z is $G = (xyz)^{1/3}$ and the **arithmetic mean** is $A = \frac{1}{3}(x + y + z)$. Use the method of Lagrange multipliers to show that $G(x, y, z) \leq A(x, y, z)$ for all x, y, z.

84. Liquid flows through a tube with length L centimeters and internal radius r centimeters. The total volume V of fluid that flows each second is related to the pressure P and the viscosity a of the fluid by the formula $V = \dfrac{\pi P r^4}{8aL}$. What is the maximum error that can occur in using this formula to compute the viscosity a, if errors of $\pm 1\%$ can be made in measuring r and L, $\pm 2\%$ in measuring V, and $\pm 3\%$ in measuring P?

85. Suppose the functions f and g have continuous partial derivatives and satisfy

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$$

These are called the *Cauchy–Riemann equations*.

a. Show that level curves of f and g intersect at right angles provided $\nabla f \neq \mathbf{0}$ and $\nabla g \neq \mathbf{0}$.

b. Assuming that the second partials of f and g are continuous, show that f and g satisfy Laplace's equations

$$f_{xx} + f_{yy} = 0 \quad \text{and} \quad g_{xx} + g_{yy} = 0$$

86. Show that if $z = f(r, \theta)$, where r and θ are defined as functions of x and y by the equations $x = r\cos\theta$, $y = r\sin\theta$, then the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ becomes $\dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r} = 0$. This is Laplace's equation in polar coordinates.

87. A circular sector of radius r and central angle θ has area $A = \frac{1}{2}r^2\theta$. Find θ for the sector of fixed area A_0 for which the perimeter of the sector is minimized. Use the method of Lagrange multipliers and assume that the minimum exists.

88. For the production function given by $Q(x, y) = x^a y^b$, where $a > 0$ and $b > 0$, show that

$$x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} = (a + b)Q$$

In particular, if $b = 1 - a$ with $0 < a < 1$,

$$x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} = Q$$

89. The diameter of the base and the height of a closed right circular cylinder are measured, and the measurements are known to have errors of at most 0.5 cm. If the diameter and height are taken to be 4 cm and 8 cm, respectively, find bounds for the propagated error in

a. the volume V of the cylinder

b. the surface area S of the cylinder

90. A right circular cone is measured and is found to have base radius $r = 40$ cm and altitude $h = 20$ cm. If it is known that each measurement is accurate to within 2%, what is the maximum percentage error in the measurement of the volume?

91. An elastic cylindrical container is filled with air so that the radius of the base is 2.02 cm and the height is 6.04 cm. If the container is deflated so that the radius of the base reduces to 2 cm and the height to 6 cm, approximately how much air has been removed? (Ignore the thickness of the container.)

92. Suppose f is a differentiable function of two variables with f_x and f_y also differentiable, and assume that f_{xx}, f_{yy}, and f_{xy} are continuous. The **second directional derivative** of f at the point (x, y) in the direction of the unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is defined by $D_{\mathbf{u}}^2 f(x, y) = D_{\mathbf{u}}[D_{\mathbf{u}} f(x, y)]$. Show that

$$D_{\mathbf{u}}^2 f(x, y) = a^2 f_{xx}(x, y) + 2ab f_{xy}(x, y) + b^2 f_{yy}(x, y)$$

93. A capsule is a cylinder of radius r and length ℓ, capped on each end by a hemisphere. Assume that the capsule dissolves in the stomach at a rate proportional to the ratio $R = S/V$, where V is the volume and S is the surface area of the capsule. Show that

$$\frac{\partial R}{\partial r} < 0 \quad \text{and} \quad \frac{\partial R}{\partial \ell} < 0$$

94. **Counterexample Problem** Find the minimum distance from the origin to the paraboloid $z = 4 - x^2 - 4y^2$. The graph is shown in Figure 11.54.

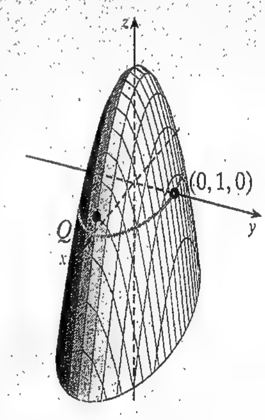


Figure 11.54

The distance from $P(x, y, z)$ to the origin is

$$d = \sqrt{x^2 + y^2 + z^2}$$

This distance will be minimized when d^2 is minimized. Thus, the function to be minimized (after replacing x^2 by $4 - 4y^2 - z$) is $D(x, y) = 4 - 4y^2 - z + y^2 + z^2 = 4 - 3y^2 + z^2 - z$. Setting the partial derivatives D_y and D_z to 0 gives the critical point $y = 0, z = 0.5$. Solving for x on the paraboloid gives the points $Q_1(\sqrt{3.5}, 0, 0.5)$ and $Q_2(-\sqrt{3.5}, 0, 0.5)$. These points are *not* minimal because $(0, 1, 0)$ is closer. Explain what is going on here. Our thanks to Herbert R. Bailey, who presented this problem in *The College Mathematics Journal* ("'Hidden' Boundaries in Constrained Max-Min Problems," May 1991, Vol. 22, p. 227).

96. It can be shown that if $z = f(x, y)$ has the least surface area of

all surfaces with a given boundary, then

$$(1+z_y^2)z_{xx} - zz_{xy} + (1+z_x^2)z_{yy} = 0$$

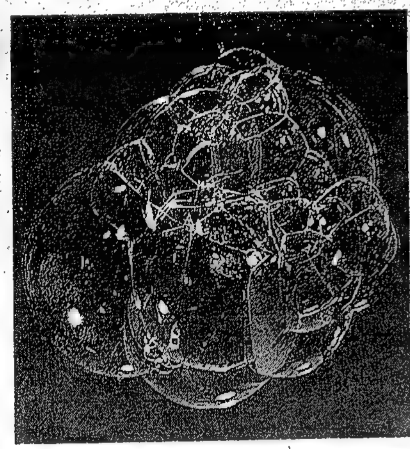
A surface satisfying such an equation is called a **minimal surface.**

a. Find constants A, B so that

$$z = \ln\left(\frac{A\cos y}{B\cos x}\right)$$

is a minimal surface.

b. Is it possible to find C and D so that $z = C\ln(\sin x) + D\ln(\sin y)$ is a minimal surface?



Soap bubbles form minimal surfaces. For an interesting discussion, see "The Geometry of Soap Films and Soap Bubbles," by Frederick J. Almgren, Jr., and Jean E. Taylor, *Scientific American*, July 1976, pp. 82–93.

97. Putnam Examination Problem Let f be a real-valued function having partial derivatives defined for $x^2 + y^2 < 1$ that satisfies $|f(x, y) \leq 1|$. Show that there exists a point (x_0, y_0) in the interior of the unit circle such that $[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 \leq 16$.

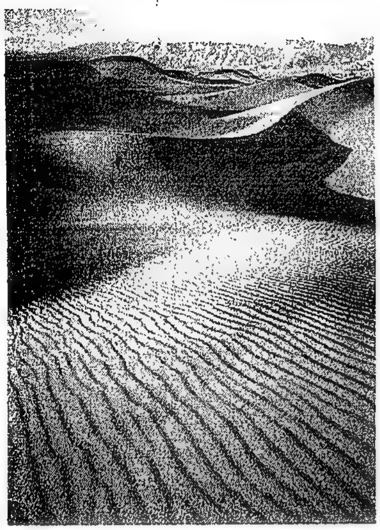
98. Putnam Examination Problem Find the smallest volume bounded by the coordinate planes and a tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

99. Putnam Examination Problem Find the shortest distance between the plane $Ax + By + Cz + 1 = 0$ and the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

Desertification



MATHEMATICS in its pure form, as arithmetic, algebra, geometry, and the applications of the analytic method, as well as mathematics applied to matter and force, or statics and dynamics, furnishes the peculiar study that gives to us, whether as children or as men, the command of nature in this its quantitative aspect; mathematics furnishes the instrument, the tool of thought, which we wield in this realm.

—W. T. Harris
Psychological Foundations of Education
(New York, 1898), p. 325.

This project is to be done in groups of three or four students. Each group will submit a single written report.

A friend of yours named Maria is studying the causes of the continuing expansion of deserts (a process known as *desertification*). She is working on a biological index of vegetation disturbance, which she has defined. By seeing how this index and other factors change through time, she hopes to discover the role played in desertification. She is studying a huge tract of land bounded by a rectangle. This piece of land surrounds a major city but does not include the city. She needs to find an economical way to calculate for this piece of land the important vegetation disturbance index $J(x, y)$.

Maria has embarked upon an ingenious approach of combining the results of photographic and radar images taken during flights over the area to calculate the index J. She is assuming that J is a smooth function. Although the flight data do not directly reveal the values of the function J, they give the rate at which the values of J change as the flights sweep over the landscape surrounding the city. Maria's staff has conducted numerous flights, and from the data she believes she has been able to find actual formulas for the rates at which J changes in the east-west and north-south directions. She has given these functions the names M and N. Thus, $M(x, y)$ is the rate at which J changes as one sweeps in the positive x-direction and $N(x, y)$ is the corresponding rate in the y-direction. Maria shows you these formulas:

$$M(x, y) = 3.4e^{x}(y - 7.8)^2 \qquad \text{and} \qquad N(x, y) = 22\sin(75 - 2xy)$$

Convince her that these two formulas cannot possibly be correct. Do this by showing her that there is a condition that the two functions M and N must satisfy if they are to be the east-west and north-south rates of change of the function J and that her formulas for M and N do not meet this condition. However, show Maria that if she can find formulas for M and N that satisfy the condition that you showed her, it is possible to find a formula for the function J from the formulas for M and N.

*Marcus S. Cohen, Edward D. Gaughan, R. Arthur Knoebel, Douglas S. Kurtz, and David J. Pengelley, "Priming the Calculus Pump: Innovations and Resources," *MAA Notes* 17(1991).

12.1

12.2

12.3

12.4

12.

12.

12

not theorems

CONTENTS

Multiple Integration

(All egs
No definition)
No theorems pfs
only statement

PREVIEW

The *single integral* $\int_a^b f(x)\,dx$ introduced in Chapter 5 has many uses, as we have seen. In this chapter, we will generalize the single integral to define *multiple* integrals in which the integrand is a function of several variables. We will find that multiple integration is used in much the same way as single integration, by "adding" small quantities to define and compute area, and volume.

PERSPECTIVE

What is the volume of a doughnut (torus)? Given the joint probability density function for the amount of time a typical shopper spends shopping at a particular store and the time spent in the checkout line, how likely is it that a shopper will spend no more than 30 minutes altogether in the store? If the temperature in a solid body is given at each point (x, y, z) and time t, what is the average temperature of the body over a particular time period? Where should a security watch tower be placed in a parking lot to ensure the most comprehensive visual coverage? We will answer these and other similar questions in this chapter using multiple integration.

12.1 Double Integration over Rectangular Regions

IN THIS SECTION definition of the double integral, properties of double integrals, volume interpretation, iterated integration, an informal argument for Fubini's theorem

DEFINITION OF THE DOUBLE INTEGRAL

Recall that in Chapter 5, we defined the definite integral of a single variable $\int_a^b f(x)\,dx$ as a limit involving Riemann sums

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k, \quad \text{where } a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

are points in a partition of the interval $[a, b]$ and x_k^* is a representative point in the subinterval $[x_{k-1}, x_k]$. We now apply the same ideas to define a definite integral of two variables $\iint_R f(x, y)\,dA$, over the rectangle $R: a \le x \le b, c \le y \le d$.

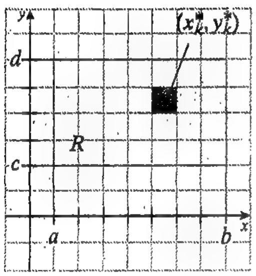


Figure 12.1 A partition of the rectangle R into mn cells showing the kth cell representative

The definition requires the ideas and notation described in the following three steps:

Step 1. Partition the interval $a \le x \le b$ into m subintervals and the interval $c \le y \le d$ into n subintervals. Using these subdivisions, partition the rectangle R into $N = mn$ **cells** (subrectangles), as shown in Figure 12.1. Call this partition P.

Step 2. Choose a representative point (x_k^*, y_k^*) from each cell in the partition of the rectangle. Form the sum

$$\sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k$$

where ΔA_k is the area of the kth representative cell. This is called the **Riemann sum** of $f(x, y)$ with respect to the partition P and cell representatives (x_k^*, y_k^*).

Step 3. To measure the size of the rectangles in the partition P, we define the **norm** $||P||$ of the partition to be the length of the longest diagonal of any rectangle in the partition. We **refine** the partition by subdividing the cells in such a way that the norm decreases.

When this process is applied to the Riemann sum and the norm decreases indefinitely to zero, we write

$$\lim_{\|P\|\to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k$$

If this limit exits, its value is called the *double integral* of f over the rectangle R.

Double Integral

If f is defined on a closed, bounded rectangular region R in the xy-plane, then the **double integral of f over R** is defined by

$$\iint_R f(x, y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k$$

provided this limit exists, in which case, f is said to be **integrable** over R.

More formally, the limit statement

$$I = \lim_{\|P\| \to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*) \Delta A_k$$

means that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| I - \sum_{k=1}^{N} f(x_k^*, y_k^*) \Delta A_k \right| < \epsilon$$

whenever $\sum_{k=1}^{N} f(x_k^*, y_k^*) \Delta A_k$ is a Riemann sum whose norm satisfies $\|P\| < \delta$. In this process, the number of cells N depends on the partition P, and as $\|P\| \to 0$, it follows that $N \to \infty$.

It can be shown that if $f(x, y)$ is continuous on a rectangle R, then it must be integrable on R, although it is also true that certain discontinuous functions are integrable as well. Moreover, it can also be shown that if the limit that defines the definite integral exists, then it is unique in the sense that the same limiting value results no matter how the partitions and subinterval representatives are chosen. We will extend the definition of the definite integral to nonrectangular regions in Section 12.2. However, issues involving the existence and uniqueness of the definite integral are generally dealt with in a course in advanced calculus.

PROPERTIES OF DOUBLE INTEGRALS

Double integrals have many of the same properties as single integrals. Three of these properties are contained in the following theorem.

THEOREM 12.1 **Properties of double integrals**

Assume that all the given integrals exist on a rectangular region R.

Linearity rule: For constants a and b,

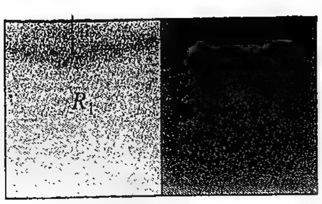
$$\iint_R [af(x, y) + bg(x, y)]\, dA = a \iint_R f(x, y)\, dA + b \iint_R g(x, y)\, dA$$

Dominance rule: If $f(x, y) \geq g(x, y)$ throughout a rectangular region R, then

$$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$$

Subdivision rule: If the rectangular region of integration R is subdivided into two subrectangles R_1 and R_2, then

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$



Region R

Proof This proof is omitted. ❑

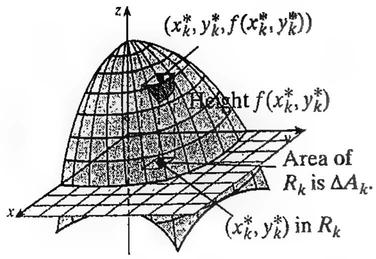


Figure 12.2 The approximating parallelepiped has volume $\Delta V_k = f(x_k^*, y_k^*)\Delta A_k$

VOLUME INTERPRETATION

If $g(x) \geq 0$ on the interval $[a, b]$, the single integral $\int_a^b g(x)\,dx$ can be interpreted as the area under the curve $y = g(x)$ over $[a, b]$. The double integral $\iint_R f(x, y)\,dA$ has a similar interpretation in terms of volume. To see this, note that if $f(x, y) \geq 0$ on the rectangular region R and we partition R, then the product $f(x_k^*, y_k^*)\Delta A_k$ is the volume of a parallelepiped (a box) with height $f(x_k^*, y_k^*)$ and base area ΔA_k, as shown in Figure 12.2.

The Riemann sum

$$\sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k$$

provides an estimate of the total volume under the surface $z = f(x, y)$ over R, and if f is continuous, we expect the approximation to improve by using more refined partitions (that is, more rectangles with smaller norm). Thus, it is natural to *define* the total volume under the surface as the limit of Riemann sums as the norm tends to 0. That is, the volume under $z = f(x, y)$ over the domain R is given by

$$V = \lim_{\|P\|\to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k = \iint_R f(x, y)\,dA$$

Double integral $\iint f(x,y)\,dA$ [illegible] volume [illegible] over R (domain)

The approximation by the Riemann sum is illustrated in Figure 12.3.

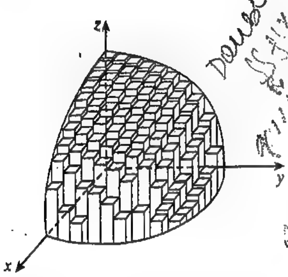


Figure 12.3 Volume approximated by rectangular parallelepipeds

EXAMPLE 1 Evaluating a double integral by relating it to a volume

Evaluate $\iint_R (2 - y)\,dA$, where R is the rectangle in the xy-plane with vertices $(0, 0)$, $(3, 0)$, $(3, 2)$, and $(0, 2)$.

Solution

Because $z = 2 - y$ satisfies $z \geq 0$ for all points in R, the value of the double integral is the same as the volume of the solid bounded above by the plane $z = 2 - y$ and below by the rectangle R. The solid is shown in Figure 12.4. Looking at it endwise, it has a triangular cross section of area B and has length 3. We use the formula $V = Bh$.

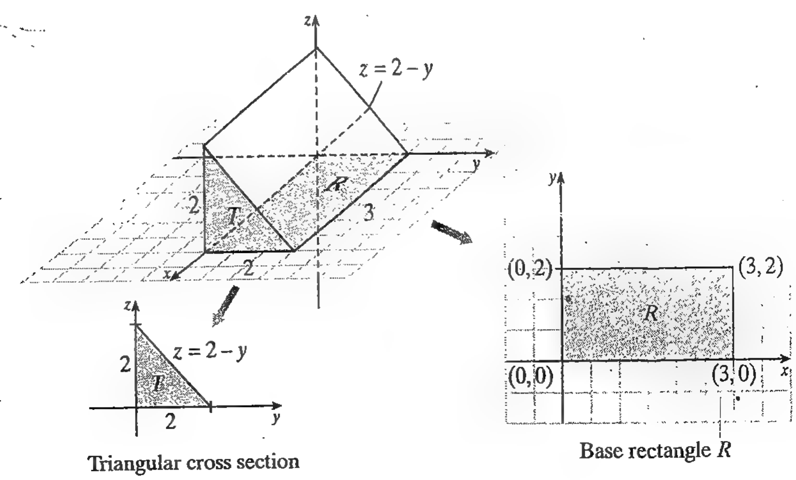


Figure 12.4 Evaluation of $\iint_R (2 - y)\,dA$ as a volume

Because the base is a triangle of side 2 and altitude 2, we have

$$V = Bh = \left[\frac{1}{2}(2)(2)\right](3) = 6$$

Therefore, the value of the integral is also 6; that is,

$$\iint_R (2 - y)\, dA = 6 \qquad \blacksquare$$

ITERATED INTEGRATION

As with single integrals, it is often not practical to evaluate a double integral even over a simple rectangular region by using the definition. Instead, we will compute double integrals by a process called **iterated integration**, which is like partial differentiation in reverse.

Suppose $f(x, y)$ is continuous over the rectangle R: $a \le x \le b$, $c \le y \le d$. Then we write $\int_c^d f(x, y)\, dy$ to denote the integral obtained by integrating $f(x, y)$ with respect to y over the interval $[c, d]$ with x held constant. The integral obtained by this partial integration is a function of x alone, $G(x) = \int_c^d f(x, y)\, dy$, which we integrate over the interval $[a, b]$ to obtain the **iterated integral**

$$\int_a^b G(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx$$

Integrate with respect to y first, keeping x constant, and then integrate with respect to x.

Similarly, if we integrate $f(x, y)$ first with respect to x over $[a, b]$ holding y constant, and then with respect to y over $[c, d]$, we obtain the iterated integral

$$\int_c^d \left[\int_a^b f(x, y)\, dx\right] dy$$

Integrate with respect to x first, keeping y constant, and then integrate with respect to y.

The following theorem, which shows how the double integral

$$\iint_R f(x, y)\, dA$$

can be evaluated in terms of iterated integrals, was proved in a more general form by the Italian mathematician Guido Fubini (1879–1943) in 1907.

THEOREM 12.2 **Fubini's theorem over a rectangular region**

If $f(x, y)$ is continuous over the rectangle R: $a \le x \le b$, $c \le y \le d$, then the double integral

$$\iint_R f(x, y)\, dA$$

may be evaluated by either iterated integral; that is,

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

➡ **What This Says** Instead of using the definition of a double integral of $f(x, y)$ over the rectangle R: $a \le x \le b$, $c \le y \le d$, evaluate either of the iterated integrals:

$$\underset{\substack{\uparrow\\ \text{Limits of } x \text{ (variable outside brackets)}}}{} \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx \quad \text{or} \quad \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

Limits of x (variable outside brackets) $\downarrow$ on the outer integral of the first form; Limits of y (variable inside brackets) $\uparrow$ on its inner integral.

Limits of y (variable outside brackets) $\downarrow$ on the outer integral of the second form; Limits of x (variable inside brackets) $\uparrow$ on its inner integral.

Note that Fubini's theorem applies only when a, b, c, and d are constants.

Proof We will provide an informal, geometric argument at the end of this section. The formal proof may be found in most advanced calculus textbooks. ❑

Let us see how Fubini's theorem can be used to evaluate double integrals. We begin by taking another look at Example 1.

EXAMPLE 2 Evaluating a double integral by using Fubini's theorem

Use an iterated integral with y-integration first to compute $\displaystyle\iint_R (2 - y)\,dA$, where R is the rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 2)$, and $(0, 2)$.

Solution

The region of integration is the rectangle $0 \le x \le 3$, $0 \le y \le 2$ (see Example 1 and Figure 12.4). Thus, by Fubini's theorem, the double integral can be evaluated by the iterated integral:

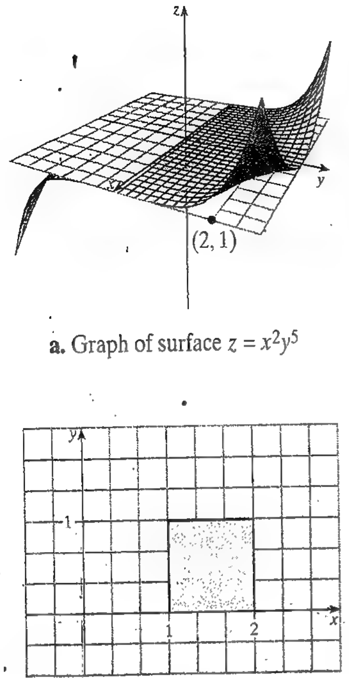
$$\begin{aligned}
\iint_R (2 - y)\,dA &= \int_0^3 \int_0^2 (2 - y)\,dy\,dx && \text{Integrate inner integral with respect to } y.\\
&= \int_0^3 \left[2y - \frac{y^2}{2} \right]\Bigg|_0^2 dx\\
&= \int_0^3 \left[4 - \frac{4}{2} - (0) \right] dx\\
&= \int_0^3 2\,dx = 2x\big|_0^3 = 6
\end{aligned}$$

which is the same as the result obtained geometrically in Example 1. ■

EXAMPLE 3 Double integral using an iterated integral

Evaluate $\displaystyle\iint_R x^2 y^5\,dA$, where R is the rectangle $1 \le x \le 2$, $0 \le y \le 1$, using an iterated integral with

a. y-integration first **b.** x-integration first



a. Graph of surface $z = x^2y^5$

b. Sketch of rectangular region

Figure 12.5 Graph of the surface over a rectangle

Solution

a. The graph of the surface $z = x^2y^5$ over the rectangle is shown in Figure 12.5a, and the rectangular region of integration in Figure 12.5b.

$$\iint_R x^2y^5\,dA = \int_1^2 \int_0^1 x^2y^5\,dy\,dx \qquad \text{Read this as } \int_1^2 \left[\int_0^1 x^2y^5\,dy\right] dx.$$

$$= \int_1^2 \left[x^2\frac{y^6}{6}\right]\Bigg|_0^1 dx$$

$$= \int_1^2 \left[x^2\left(\frac{1}{6} - \frac{0}{6}\right)\right] dx = \frac{x^3}{18}\Bigg|_1^2 = \frac{8}{18} - \frac{1}{18} = \frac{7}{18}$$

b.

$$\iint_R x^2y^5\,dA = \int_0^1 \int_1^2 x^2y^5\,dx\,dy = \int_0^1 y^5\left[\frac{x^3}{3}\right]\Bigg|_1^2 dy$$

$$= \int_0^1 \left[y^5\left(\frac{8}{3} - \frac{1}{3}\right)\right] dy = \frac{7y^6}{18}\Bigg|_0^1 = \frac{7}{18} - \frac{0}{18} = \frac{7}{18} \quad \blacksquare$$

TECHNOLOGY NOTE

Using technology for multiple integrals offers no special difficulties. You simply integrate with respect to one variable and then with respect to the other variable. You must be careful, however, to input properly the correct limits of integration for each integral. Notice that for $\int_1^2 \int_0^1 x^2y^2\,dy\,dx$, most calculators and software programs require the following syntax: *integration operator, function, variable of integration, lower limit of integration, upper limit of integration.*

$$\int\Big(\underbrace{\Big(\int(\underbrace{x^2y^5}_{\textit{function}}),\ \underset{\substack{\uparrow\\ \textit{inner variable}\\ \textit{of integration}}}{y},\ \overbrace{0,1}^{\substack{\textit{inner limits}\\ \textit{of integration}}}\Big)}_{\substack{\text{"inside" integral is the function}\\ \text{for the "outside" integration}}},\ \underset{\substack{\uparrow\\ \textit{outer variable}\\ \textit{of integration}}}{x},\ \overbrace{1,2}^{\substack{\textit{outer limits}\\ \textit{of integration}}}\Big)$$

The result of this operation is shown in Figure 12.6 (compare with Example 3).

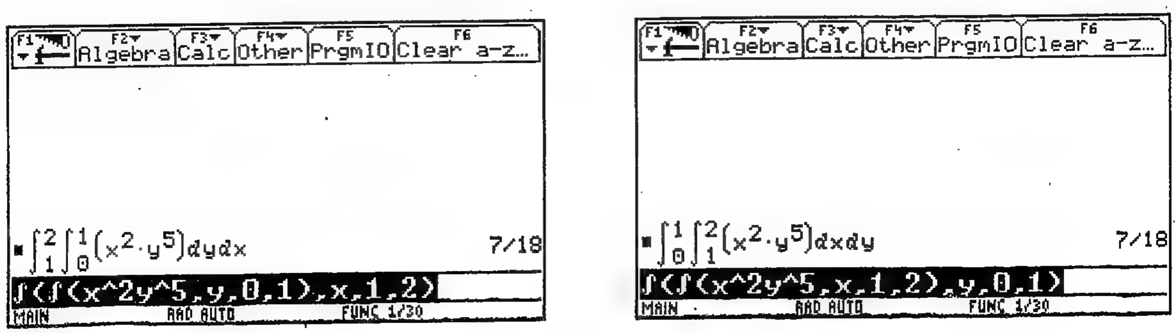


Figure 12.6 Calculator output for Example 3

In many problems, the order of integration in a double integral is largely a matter of personal choice, but occasionally, choosing the order correctly makes the difference between a straightforward evaluation and one that is either difficult or impossible. Consider the following example.

EXAMPLE 4 Choosing the order of integration for a double integral

Evaluate $\iint_R x \cos xy \, dA$ for R: $0 \le x \le \frac{\pi}{2}$, $0 \le y \le 1$.

Solution

Suppose we integrate with respect to x first:

$$\int_0^1 \left[\int_0^{\pi/2} x \cos xy \, dx \right] dy$$

The inner integral requires integration by parts. However, integrating with respect to y first is much simpler:

$$\int_0^{\pi/2} \left[\int_0^1 x \cos xy \, dy \right] dx = \int_0^{\pi/2} \left[\frac{x \sin xy}{x} \right]_0^1 dx$$

$$= \int_0^{\pi/2} (\sin x - \sin 0) \, dx = -\cos x \Big|_0^{\pi/2} = 1 \quad \blacksquare$$

AN INFORMAL ARGUMENT FOR FUBINI'S THEOREM

We can make Fubini's theorem plausible with a geometric argument in the case where $f(x, y) \ge 0$ on R. If $\iint_R f(x, y) \, dA$ is defined on a rectangle R: $a \le x \le b$, $c \le y \le d$, it represents the volume of the solid D bounded above by the surface $z = f(x, y)$ and below by the rectangle R. If $A(y_k^*)$ is the cross-sectional area perpendicular to the y-axis at the point y_k^*, then $A(y_k^*)\Delta y_k$ represents the volume of a "slab" that approximates the volume of part of the solid D, as shown in Figure 12.7.

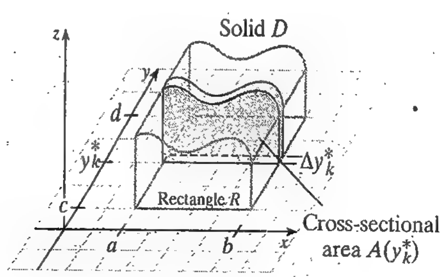


Figure 12.7 Cross-sectional volume parallel to the xz-plane

By using a limit to "add up" all such approximating volumes, we obtain the volume, V, of the entire solid D; that is,

$$\iint_R f(x, y) \, dA = V = \lim_{\|P\| \to 0} \sum_{k=1}^{N} A(y_k^*) \Delta y_k$$

The limit on the right is just the integral of $A(y)$ over the interval $c \le y \le d$, where $A(y)$ is the area of a cross section with fixed y. In Chapter 5, we found that the area $A(y)$ can be computed by the integral

$$A(y) = \int_a^b f(x, y)\,dx \qquad \text{Integration with respect to } x \text{ (} y \text{ is fixed)}$$

We can now make this substitution for $A(y)$ to obtain

$$\begin{aligned}\iint_R f(x, y)\,dA = V &= \lim_{\|P\|\to 0} \sum_{k=1}^{N} A(y_k^*)\Delta y_k \\ &= \int_c^d A(y)\,dy \\ &= \int_c^d \underbrace{\left[\int_a^b f(x, y)\,dx\right]}_{A(y)} dy \qquad \text{Substitution}\end{aligned}$$

The fact that

$$\iint_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

can be justified in a similar fashion (you are asked to do this in Problem 21). Thus, we have

$$\int_c^d \left[\int_a^b f(x, y)\,dx\right] dy = \iint_R f(x, y)\,dA = \int_a^b \left[\int_c^d f(x, y)\,dy\right] dx$$

Guideline Qs 1–18

12.1 PROBLEM SET

HW (Pg160–Pg173)

A *In Problems 1–2, evaluate the iterated integrals.*

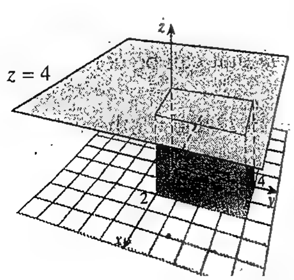
1. $\int_0^2 \int_0^1 (x^2 + xy + y^2)\,dy\,dx$

2. $\int_1^2 \int_0^{\pi} x \cos y\,dy\,dx$

Use an appropriate volume formula to evaluate the double integral given in Problems 3–8.

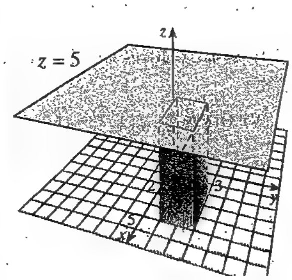
3. $\iint_R 4\,dA;$

$R: 0 \le x \le 2, 0 \le y \le 4$



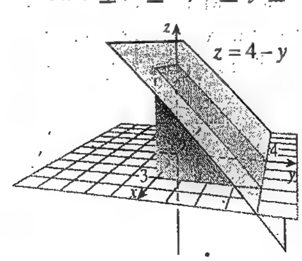
4. $\iint_R 5\,dA;$

$R: 2 \le x \le 5, 1 \le y \le 3$



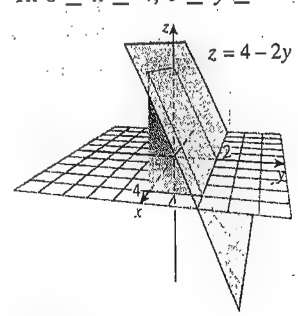
5. $\iint_R (4 - y)\,dA;$

$R: 0 \le x \le 3, 0 \le y \le 4$



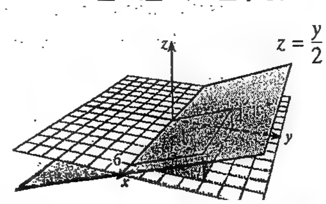
6. $\iint_R (4 - 2y)\,dA;$

$R: 0 \le x \le 4, 0 \le y \le 2$



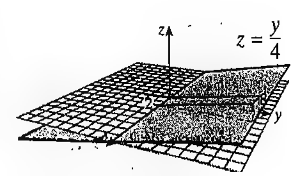
7. $\iint_R \frac{y}{2}\,dA;$

$R: 0 \le x \le 6, 0 \le y \le 4$



8. $\iint_R \frac{y}{4}\,dA;$

$R: 0 \le x \le 2, 0 \le y \le 8$



Use iterated integration to compute the double integrals in Problems 9–12 over the specified rectangle.

9. $\iint\limits_R x^2 y\,dA$; R: $1 \le x \le 2, 0 \le y \le 1$

10. $\iint\limits_R \dfrac{2xy\,dA}{x^2+1}$; R: $0 \le x \le 1, 1 \le y \le 3$

11. $\iint\limits_R \sin(x+y)\,dA$; R: $0 \le x \le \frac{\pi}{4}, 0 \le y \le \frac{\pi}{2}$

12. $\iint\limits_R x \sin xy\,dA$, R: $0 \le x \le \pi, 0 \le y \le 1$

Find the volume of the solid bounded below by the rectangle R in the xy-plane and above by the graph of $z = f(x, y)$ in Problems 13–14.

13. $f(x, y) = 2x + 3y$; R: $0 \le x \le 1, 0 \le y \le 2$

14. $f(x, y) = \sqrt{xy}$; R: $0 \le x \le 1, 0 \le y \le 4$

B *In Problems 15–17, evaluate the integral. Note that one order of integration may be considerably easier than the other.*

15. Compute $\iint\limits_R x\sqrt{1-x^2}\,e^{3y}\,dA$, where R is the rectangle $0 \le x \le 1, 0 \le y \le 2$.

16. Compute (correct to the nearest hundredth) (2 decimal places) $\iint\limits_R \dfrac{xy}{x^2+y^2}\,dA$, where R is the rectangle $1 \le x \le 3, 1 \le y \le 2$.

17. Evaluate $\iint\limits_R xe^{xy}\,dA$, where R is the rectangle $0 \le x \le 1$, $1 \le y \le 2$.

18. Explain why $\iint\limits_R (4 - x^2 - y^2)\,dA > 2$, where R is the rectangular domain in the plane given by $0 \le x \le 1, 0 \le y \le 1$.

19. **HISTORICAL QUEST** Guido Fubini taught at the Institute for Advanced Study in Princeton. He was nicknamed the "Little Giant," because of his small body but large mind. Even though the conclusion of Fubini's theorem was known for a long time and successfully applied in various instances, it was not satisfactorily proved in a general setting until 1907. His most important work was in differential projective geometry. In 1938 he was forced to leave Italy because of the Fascist government, and he immigrated to the United States.

GUIDO FUBINI
1879–1943

C 20. Let f be a function with continuous second partial derivatives on a rectangular domain R with vertices (x_1, y_1), (x_1, y_2), (x_2, y_2), and (x_2, y_1), where $x_1 < x_2$ and $y_1 < y_2$. Use the fundamental theorem of calculus to show that

$$\iint\limits_R \frac{\partial^2 f}{\partial y \partial x}\,dA = f(x_1, y_1) - f(x_2, y_1) + f(x_2, y_2) - f(x_1, y_2)$$

21. Let f be a continuous function defined on the rectangle R: $a \le x \le b$, $c \le y \le d$. Show that

$$\iint\limits_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

Hint: Modify the argument given in the text by taking cross-sectional areas perpendicular to the x-axis.

22. **Counterexample Problem** Show that the iterated integrals

$$\int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}\,dy\,dx \quad \text{and} \quad \int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}\,dx\,dy$$

have different values. Why does this not contradict Fubini's theorem?

23. **Exploration Problem** You want to evaluate

$$\iint\limits_D y \sin(xy) \sin^2(\pi y)\,dA$$

over the rectangle D: $0 \le x \le \pi, 0 \le y \le \frac{1}{2}$. Which order of integration is easier?

12.2 Double Integration over Nonrectangular Regions

IN THIS SECTION double integrals over Type I and Type II regions, more on area and volume, choosing the order of integration in a double integral

Let $f(x, y)$ be a function that is continuous on the region D that can be contained in a rectangle R. (See Figure 12.8.)

Define the function $F(x, y)$ on R as $f(x, y)$ if (x, y) is in D and 0 otherwise. That is,

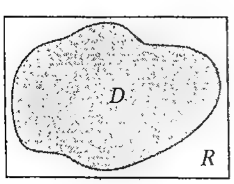


Figure 12.8 The region D is bounded by a rectangle R

$$F(x, y) = \begin{cases} f(x, y) & \text{for } (x, y) \text{ in } D \\ 0 & \text{for } (x, y) \text{ in } R, \text{ but not in } D \end{cases}$$

Then, if F is integrable over R, we say that f is **integrable over D**, and the **double integral of f over D** is defined as

$$\iint_D f(x, y)\,dA = \iint_R F(x, y)\,dA$$

The function $F(x, y)$ may have discontinuities on the boundary of D, but if $f(x, y)$ is continuous on D and the boundary of D is fairly "well behaved," then it can be shown that $\iint_R F(x, y)\,dA$ exists and hence that $\iint_D f(x, y)\,dA$ exists. This procedure is valid for the type I and type II regions we discuss next, although a general notion of what constitutes a "well-behaved" boundary is a topic for advanced calculus.

DOUBLE INTEGRALS OVER TYPE I AND TYPE II REGIONS

A type I, or vertically simple region, D, in the plane is a region that can be described by the inequalities

$$\textbf{type I} \qquad D_1\colon a \le x \le b, \quad g_1(x) \le y \le g_2(x)$$

where $g_1(x)$ and $g_2(x)$ are continuous functions of x on $[a, b]$.

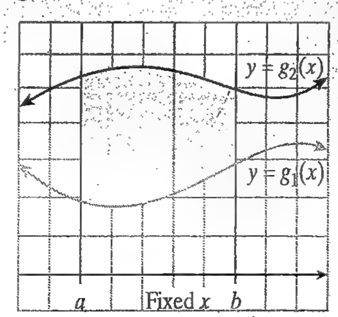
Likewise, a type II, or horizontally simple region, D, is one that can be described by the inequalities

$$\textbf{type II} \qquad D_2\colon c \le y \le d, \quad h_1(y) \le x \le h_2(y)$$

where $h_1(y)$ and $h_2(y)$ are continuous functions of y on $[c, d]$. Vertically and horizontally simple regions are illustrated in the following box.

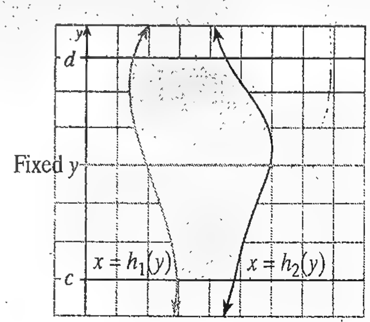
Type I Region (vertically simple)

A **type I** region is the set of all points (x, y) such that for each fixed x between $x = a$ and $x = b$, the vertical line segment $g_1(x) \le y \le g_2(x)$ lies in the region.



Type II Region (horizontally simple)

A **type II** region is the set of all points (x, y) such that for each fixed y between $y = c$ and $y = d$, the horizontal line segment $h_1(y) \le x \le h_2(y)$ lies in the region.



To evaluate a double integral $\int_{D_1} f(x, y)\,dA$ when D_1 is a type I region, first note that D_1 is contained in the rectangle R: $a \le x \le b$, $c \le y \le d$. Then, if

$$F(x, y) = \begin{cases} f(x, y) & \text{for } (x, y) \text{ in } D_1 \\ 0 & \text{for } (x, y) \text{ in } R, \text{ but not in } D_1 \end{cases}$$

we have

$$\int_c^d F(x, y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy$$

because for each fixed x in the interval $[a, b]$, $F(x, y) = 0$ if $y < g_1(x)$ and also if $y > g_2(x)$ and $F(x, y) = f(x, y)$ for $g_1(x) \leq y \leq g_2(x)$. Therefore,

$$\iint_{D_1} f(x, y)\,dA = \iint_R F(x, y)\,dA = \int_a^b \left[\int_c^d F(x, y)\,dy\right] dx$$
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

Similarly, if D_2 is a type II region, then

$$\iint_{D_2} f(x, y)\,dA = \int_c^d \left[\int_a^b F(x, y)\,dx\right] dy$$
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy$$

($\because$ for each $y \in [c, d]$, $F(x, y) = 0$ if $x < h_1(y)$ & if $x > h_2(y)$ and $F(x, y) = f(x, y)$ for $h_1(y) \leq x \leq h_2(y)$.)

These observations are summarized in the following theorem.

THEOREM 12.3 Fubini's theorem for nonrectangular regions

If D_1 is a type I region, then

TYPE I (vertically simple): x fixed, y varies (form $dy\,dx$)

$$\iint_{D_1} f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

whenever both integrals exist. Similarly, for a type II region, D_2,

TYPE II (horizontally simple): y fixed, x varies (form $dx\,dy$)

$$\iint_{D_2} f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy$$

whenever both integrals exist.

Proof This proof is found in most advanced calculus textbooks. ❑

EXAMPLE 1 Evaluation of a double integral

Evaluate $\displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} 160xy^3\,dy\,dx.$

Solution

$$\int_0^1 \int_{x^2}^{\sqrt{x}} 160xy^3\,dy\,dx = \int_0^1 \left[40xy^4\Big|_{y=x^2}^{y=\sqrt{x}}\right] dx \qquad \text{Since } x \text{ is treated as a constant}$$

$$= \int_0^1 [40x\left(\sqrt{x}\right)^4 - 40x(x^2)^4]\,dx$$

$$= \int_0^1 [40x^3 - 40x^9]\,dx$$

$$= \left[10x^4 - 4x^{10}\right]\Big|_0^1$$

$$= 6$$

■

When using Fubini's theorem for nonrectangular regions, it helps to sketch the region of integration D and to find equations for all boundary curves of D. Such a sketch often provides the information needed to determine whether D is a type I or type II region (or neither, or both) and to set up the limits of integration of an iterated integral.

EXAMPLE 2 Double integral over a triangular region

Let T be the triangular region enclosed by the lines $y = 0$, $y = 2x$, and $x = 1$. Evaluate the double integral

$$\iint_T (x + y)\,dA$$

using an iterated integral with

a. y-integration first **b.** x-integration first

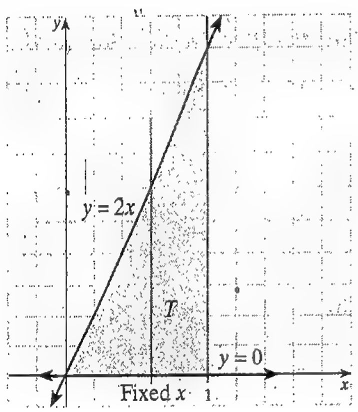


Figure 12.9 For each fixed x ($0 \le x \le 1$), y varies from $y = 0$ to $y = 2x$.

Solution

a. To set up the limits of integration in the iterated integral, we draw the graph as shown in Figure 12.9 and note that for fixed x, the variable y varies from $y = 0$ (the x-axis) to the line $y = 2x$. These are the limits of integration for the inner integral (with respect to y first). The outer limits of integration are the numerical limits of integration for x; that is, x varies between $x = 0$ and $x = 1$.

$$\iint_T (x + y)\,dA = \int_0^1 \int_0^{2x} (x + y)\,dy\,dx = \int_0^1 \left[xy + \tfrac{1}{2}y^2\right]\Big|_{y=0}^{y=2x} dx$$

$$= \int_0^1 \left[x(2x) + \tfrac{1}{2}(2x)^2 - \left(x(0) + \tfrac{1}{2}(0)^2\right)\right] dx$$

$$= \int_0^1 4x^2\,dx = \tfrac{4}{3}\,x^3\Big|_{x=0}^{x=1} = \tfrac{4}{3}$$

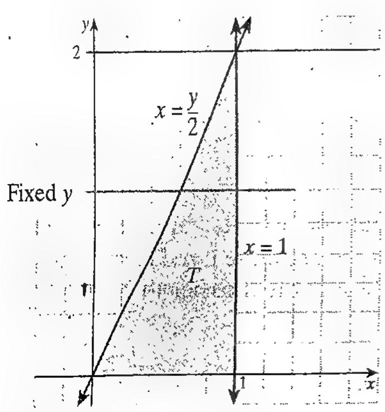


Figure 12.10 For each fixed y ($0 \le y \le 2$), x varies from $y/2$ to 1

b. Reversing the order of integration, we see from Figure 12.10 that for each fixed y, the variable x varies (left to right) from the line $x = y/2$ to the vertical line $x = 1$. The outer limits of integration are for y as y varies from $y = 0$ to $y = 2$.

$$\iint_T (x + y)\,dA = \int_0^2 \int_{y/2}^1 (x + y)\,dx\,dy$$

$$= \int_0^2 \left[\frac{1}{2}x^2 + xy\right]\Big|_{x=y/2}^{x=1} dy = \int_0^2 \left[\frac{1}{2} + y - \frac{y^2}{8} - \frac{y^2}{2}\right] dy$$

$$= \left[\frac{y}{2} + \frac{y^2}{2} - \frac{5y^3}{24}\right]\Big|_{y=0}^{y=2} = \left[1 + 2 - \frac{5(8)}{24}\right] - [0] = \frac{4}{3}$$

■

MORE ON AREA AND VOLUME

Even though we can find the area between curves with single integrals, it is often easier to compute area using a double integral. If $f(x, y) \geq 0$ over a region D in the xy-plane, then $\iint\limits_D f(x, y)\, dA$ gives the **volume of the solid** bounded above by the surface $z = f(x, y)$ and below by the region D. In the special case where $f(x, y) = 1$, we have $\iint\limits_D 1\, dA = \text{AREA OF } D$.

The Double Integral as Area and Volume

The **area** of the region D in the xy-plane is given by

$$A = \iint\limits_D dA$$

If f is continuous and $f(x, y) \geq 0$ on the region D, the **volume** of the solid under the surface $z = f(x, y)$ above the region D is given by

$$V = \iint\limits_D f(x, y)\, dA$$

EXAMPLE 3 Area of a region in the xy-plane using a double integral

Find the area of the region D between $y = \cos x$ and $y = \sin x$ over the interval $0 \leq x \leq \frac{\pi}{4}$ using

a. single integral **b.** a double integral

Solution

a. The graph is shown in Figure 12.11. We find that

$$\int_0^{\pi/4} (\cos x - \sin x)\, dx = [\sin x + \cos x]\Big|_0^{\pi/4} = \sqrt{2} - 1$$

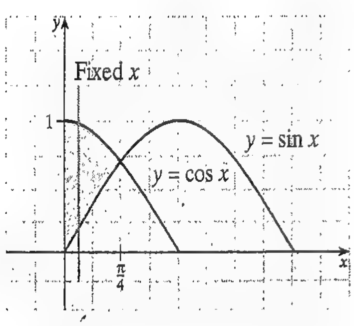


Figure 12.11 The area of the region between $y = \cos x$ and $y = \sin x$

b. $$A = \iint\limits_D dA = \int_0^{\pi/4} \int_{\sin x}^{\cos x} 1\, dy\, dx = \int_0^{\pi/4} [y]\Big|_{y=\sin x}^{y=\cos x} dx$$

$$= \int_0^{\pi/4} [\cos x - \sin x]\, dx = \sqrt{2} - 1$$

The area $\sqrt{2} - 1 \approx 0.41$ square unit. ■

In comparing the single and double integral solutions for area in Example 3, you might ask, "Why bother with the double integral, because it reduces to the single integral case after one step?" The answer is that it is often easier to begin with the double integral

$$A = \iint\limits_D dA$$

and then let the *evaluation* lead to the proper form.

EXAMPLE 4 Volume using a double integral

Find the volume of the solid bounded above by the plane $z = y$ and below in the xy-plane by the part of the disk $x^2 + y^2 \le 1$ in the first quadrant.

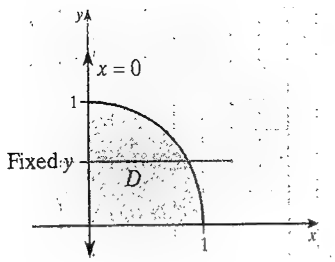


Figure 12.12 The quarter disk $x^2 + y^2 \le 1, x \ge 0, y \ge 0$

Solution

The projection in the xy-plane is shown in Figure 12.12. We can regard D as either a type I or type II region, and because we worked with a type I (vertical) region in Example 3, we will use a type II (horizontal) region for this example. Accordingly, note that for each fixed number y between 0 and 1, x varies between $x = 0$ on the left and $x = \sqrt{1 - y^2}$ on the right. Thus,

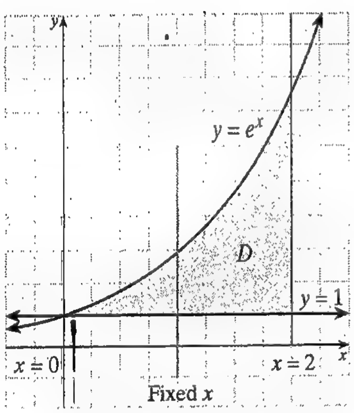
$$
\begin{aligned}
V &= \iint\limits_D f(x, y)\, dA \\
&= \int_0^1 \int_0^{\sqrt{1-y^2}} y\, dx\, dy && f(x, y) = y \text{ is given; } dA = dx\, dy. \\
&= \int_0^1 [yx]\Big|_{x=0}^{x=\sqrt{1-y^2}} dy \\
&= \int_0^1 y\sqrt{1 - y^2}\, dy && \text{Let } u = 1 - y^2 \text{ to integrate.} \\
&= \left[-\frac{1}{3}(1 - y^2)^{3/2}\right]\Big|_{y=0}^{y=1} = \frac{1}{3}
\end{aligned}
$$

■

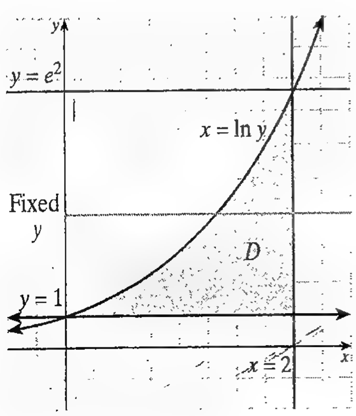
The volume is $\frac{1}{3}$ cubic unit.

CHOOSING THE ORDER OF INTEGRATION IN A DOUBLE INTEGRAL

Often a region D is both vertically and horizontally simple, and to evaluate an integral $\int_D f(x, y)\, dA$ you have a choice between performing x-integration before y-integration, or vice versa. In the following example, you are given one order of integration and are asked to reverse the order of integration.



a. D as a type I region; y varies from 1 to e^x



b. D as a type II region; x varies from $\ln y$ to 2

Figure 12.13 The region of integration for Example 5

EXAMPLE 5 Reversing order of integration in a double integral

Reverse the order of integration in the iterated integral

$$\int_0^2 \int_1^{e^x} f(x, y)\, dy\, dx$$

Solution

Draw the region D by looking at the limits of integration for both x and y in the double integral. For this example, we see that the y-integration is done first, so D is a type I region. The inner limits are

$$y = e^x \quad \text{(top curve)} \qquad \text{and} \qquad y = 1 \quad \text{(bottom curve)}$$

These are shown in Figure 12.13a. Next, draw the appropriate limits of integration for x:

$$x = 0 \quad \text{(left point)} \qquad \text{and} \qquad x = 2 \quad \text{(right point)}$$

The vertical lines $x = 0$ and $x = 2$ are also drawn in Figure 12.13a.

To reverse the order of integration, we need to regard D as a type II region (Figure 12.13b). Note that the region varies from $y = 1$ to $y = e^2$ (corresponding to

where $y = e^x$ intersects $x = 0$ and $x = 2$, respectively). For each fixed y between 1 and e^2, the region extends from the curve $x = \ln y$ (that is, $y = e^x$) on the left to the line $x = 2$ on the right. Thus, reversing the order of integration, we find that the given integral becomes

$$\int_1^{e^2}\int_{\ln y}^{2} f(x, y)\,dx\,dy$$

The two different ways of representing the integral in this example may be summarized as follows:

Type I: x fixed (vertically simple) — y-integration first; varies from $y = 1$ to $y = e^x$

Type II: y fixed (horizontally simple) — x-integration first; varies from $x = \ln y$ to $x = 2$

$$\int_0^2\int_1^{e^x} f(x, y)\,dy\,dx \quad = \quad \int_1^{e^2}\int_{\ln y}^{2} f(x, y)\,dx\,dy$$

x varies from 0 to 2 $\qquad$ y varies from 1 to e^2 ■

EXAMPLE 6 Choosing the order of integration

The region D bounded by the parabola $y = x^2 - 2$ and the line $y = x$ is both vertically and horizontally simple. To find the area of D, would you prefer to use a type I or a type II description?

Solution

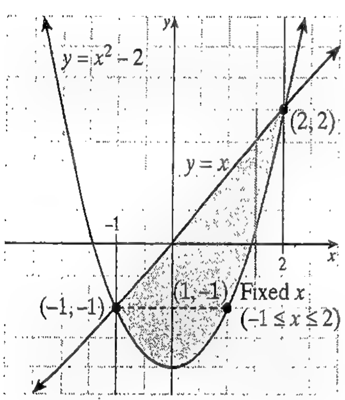
The parabola and the line intersect where

$$\begin{aligned} x^2 - 2 &= x \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= 2, -1 \end{aligned}$$

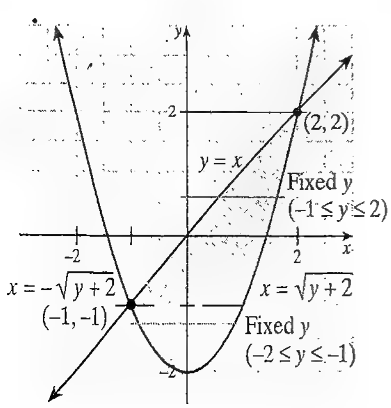
Thus, they intersect at $(2, 2)$ and $(-1, -1)$. The graph of D is shown in Figure 12.14.

As a type I region, D can be described as the set of all points (x, y) such that for each fixed x in $[-1, 2]$, y varies from $x^2 - 2$ to x (see Figure 12.14). The area, A, of D is given by

$$\begin{aligned} A &= \int_{-1}^{2}\int_{x^2-2}^{x} dy\,dx \\ &= \int_{-1}^{2} [x - (x^2 - 2)]\,dx \\ &= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]\Bigg|_{-1}^{2} \\ &= \frac{9}{2} \end{aligned}$$



a. Type I description



b. Type II description

Figure 12.14 The region between the parabola $y = x^2 - 2$ and the line $y = x$.

If, however, D is regarded as a type II region, it is necessary to split up the description into two parts (see Figure 12.14b):

The set of all (x, y) such that for each fixed y between -1 and 2, x varies from the line $x = y$ to the right branch of the parabola $x = \sqrt{y + 2}$;

and

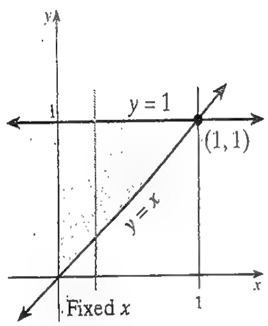
The set of all (x, y) such that for each fixed y between -2 and -1, x varies from the left branch of the parabola $x = -\sqrt{y+2}$ to the right branch $x = \sqrt{y+2}$.

Thus, the area of D is given by the sum of the integrals

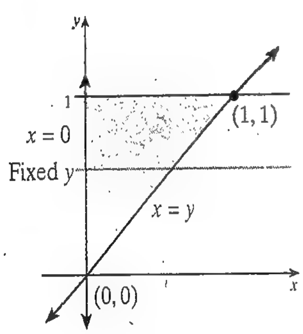
$$A = \int_{-2}^{-1} \int_{-\sqrt{y+2}}^{\sqrt{y+2}} dx\,dy + \int_{-1}^{2} \int_{y}^{\sqrt{y+2}} dx\,dy$$

Clearly, it is preferable to use the type I representation, although both methods give the same result (you might wish to verify this fact). ■

As illustrated in Example 6, the shape of the region of integration D may determine which order of integration is more suitable for a given integral. However, our final example of this section shows that the integrand also plays a role in determining which order of integration is preferable.



a. y-integration first



b. x-integration first

Figure 12.15 The region of integration for Example 7

EXAMPLE 7 Evaluating a double integral by reversing the order

Evaluate $\displaystyle\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$

Solution

We cannot evaluate the integral in the given order (y-integration first) because the integrand e^{y^2} has no elementary antiderivative. We will evaluate the integral by reversing the order of integration. The region of integration is sketched in Figure 12.15a. Note that for any fixed x between 0 and 1, y varies from x to 1.

To reverse the order of integration, observe that for each fixed y between 0 and 1, x varies from 0 to y, as shown in Figure 12.15b.

$$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx = \int_0^1 \int_0^y e^{y^2}\,dx\,dy$$

$$= \int_0^1 xe^{y^2}\Big|_{x=0}^{x=y} dy = \int_0^1 ye^{y^2}\,dy \qquad \textit{Let } u = y^2.$$

$$= \left[\frac{1}{2}e^{y^2}\right]\Bigg|_0^1 = \frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e-1)$$ ■

12.2 PROBLEM SET (pg 182-198)

#1-17

CW pg 98

A *Sketch the region of integration in Problems 1–4, and compute the double integral (either in the order of integration given or with the order reversed).*

1. $\displaystyle\int_0^{2\sqrt{3}} \int_{y^2/4}^{\sqrt{12-y^2}} dx\,dy$

2. $\displaystyle\int_0^1 \int_{x^3}^1 (x+y^2)\,dy\,dx$

3. $\displaystyle\int_{-1}^1 \int_{-1}^x (3x+2y)\,dy\,dx$

4. $\displaystyle\int_0^{\pi/2} \int_0^{\sin x} e^y \cos x\,dy\,dx$

Evaluate the double integrals in Problems 5–6

5. $\displaystyle\int_0^1 \int_0^{y^3} e^{x/y}\,dx\,dy$

6. $\displaystyle\int_0^1 \int_{\tan^{-1} y}^{\pi/4} \sec x\,dx\,dy$

Evaluate the double integral given in Problems 7–9 for the specified region of integration D.

7. $\displaystyle\iint_D (x+y)\,dA$; D is the triangle with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$.

8. $\displaystyle\iint_D (2y-x)\,dA$; D is the region bounded by $y = x^2$ and $y = 2x$.

9. $\iint\limits_D \frac{\quad}{y^2+1}$; D is the triangle bounded by $x=2y$, $y=-x$, and $y=2$.

Sketch the region of integration in Problems 10–11, and then compute the integral in two ways:

a. with the given order of integration, and
b. with the order of integration reversed.

10. $\displaystyle\int_0^4 \int_0^{4-x} xy\,dy\,dx$ 11. $\displaystyle\int_0^1 \int_x^{2x} e^{y-x}\,dy\,dx$

Sketch the region of integration in Problems 12–13, and write an equivalent integral with the order of integration reversed.

12. $\displaystyle\int_0^1 \int_0^{2y} f(x,y)\,dx\,dy$ 13. $\displaystyle\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$

B *Set up a double integral for the volume of the solid region described in Problems 14–17.*

14. The tetrahedron that lies in the first octant and is bounded by the coordinate planes and the plane $z = 7 - 3x - 2y$

15. The solid bounded above by the paraboloid $z = 6 - 2x^2 - 3y^2$ and below by the plane $z = 0$

16. The solid bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

17. The solid bounded above by the plane $z = 2 - 3x - 5y$ and below by the region shown in Figure 12.16.

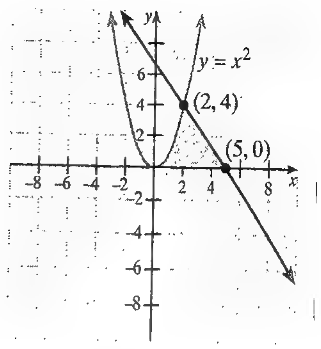


Figure 12.16 Region for Problem 17

12.3 Double Integrals in Polar Coordinates

(read pg 260 table of polar curves from Strauss (I))

IN THIS SECTION change of variables to polar form, area and volume in polar form

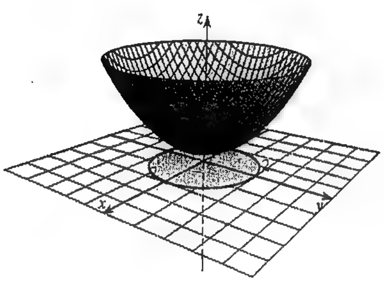


Figure 12.17 Graph of the surface $f(x, y) = x^2 + y^2 + 1$ along with the region R shown in the xy-plane

In general, changing variables in a double integral is more complicated than in a single integral. In this section, we focus attention on using polar coordinates in a double integral, and in Section 12.6, we examine changing variables from a more general standpoint.

CHANGE OF VARIABLES TO POLAR FORM

Polar coordinates are used in double integrals primarily when the integrand or the region of integration (or both) have relatively simple polar descriptions. As a preview of the ideas we plan to explore, consider Figure 12.17 and let us examine the double integral

$$\iint\limits_R (x^2 + y^2 + 1)\,dA$$

where R is the region (disk) in the xy-plane bounded by the circle $x^2 + y^2 = 4$.

Interpreting R as a vertically simple region (type I), we see that for each fixed x between -2 and 2, y varies from the lower boundary semicircle with equation $y = -\sqrt{4-x^2}$ to the upper semicircle $y = \sqrt{4-x^2}$, as shown in Figure 12.18a. Using the type I description, we have

$$\iint_R (x^2+y^2+1)\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2+y^2+1)\,dy\,dx$$

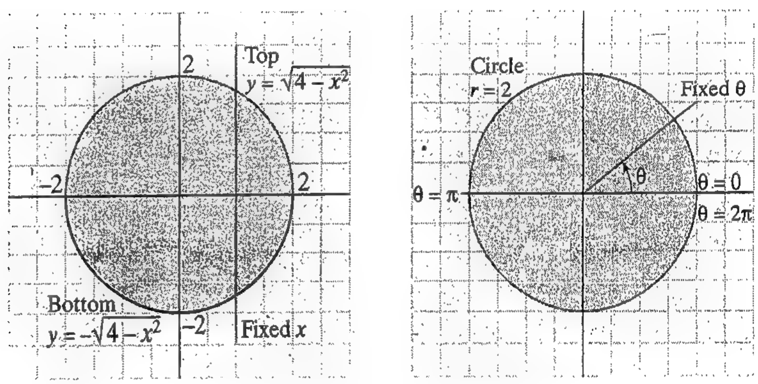
The iterated integral on the right is difficult to evaluate, but both the integrand and the domain of integration can be represented quite simply in terms of polar coordinates. Specifically, using the polar conversion formulas

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2+y^2} \qquad \tan\theta = \frac{y}{x}$$

we find that the integrand $f(x, y) = x^2 + y^2 + 1$ can be rewritten

$$f(r\cos\theta,\, r\sin\theta) = (r\cos\theta)^2 + (r\sin\theta)^2 + 1 = r^2 + 1$$

and the region of integration R is just the interior of the circle $r = 2$. Thus, R can be described as the set of all points (r, θ) so that for each fixed angle θ between 0 and 2π, r varies from the origin ($r = 0$) to the circle $r = 2$, as shown in Figure 12.18b.



a. Type I description:
For fixed x between −2 and 2
y varies from $y = -\sqrt{4-x^2}$
to $y = \sqrt{4-x^2}$

b. Polar description:
For fixed θ between 0 and 2π,
r varies from 0 to 2.

Figure 12.18 Two interpretations of a region R

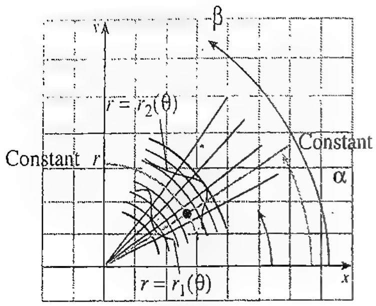
But what is the differential of integration dA in polar coordinates? Can we simply substitute "$dr\,d\theta$" for dA and perform the integration with respect to r and θ? The answer is no, and the correct formula for expressing a given double integral in polar form is given in the following theorem.

no proof

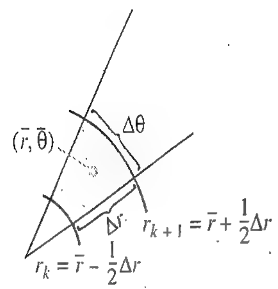
***THEOREM 12.4* Double integral in polar coordinates**

If f is continuous in the polar region D described by $r_1(\theta) \le r \le r_2(\theta)$ ($r_1(\theta) \ge 0$, $r_2(\theta) \ge 0$), $\alpha \le \theta \le \beta$ ($0 \le \beta - \alpha \le 2\pi$), then

$$\iint_D f(r,\theta)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$



a. A partition of a region into polar coordinates



b. A typical polar rectangle in the partition

Figure 12.19 A polar rectangle

➡ **What This Says** The procedure for changing from a Cartesian integral

$$\iint_R f(x, y)\, dA$$

into a polar integral requires two steps. First, substitute $x = r\cos\theta$ and $y = r\sin\theta$, $dxdy = rdrd\theta$ into the Cartesian integral. Then convert the region of integration R to polar form D. Thus,

$$\iint_R f(x, y)\, dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Proof A polar region described by $r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$ can be subdivided into **polar rectangles**. A typical polar rectangle is shown in Figure 12.19.

We begin by subdividing the region of integration into polar rectangles. Next we pick an arbitrary polar-form point (r_k^*, θ_k^*) in each polar rectangle in the partition and then take the limit of an appropriate Riemann sum

$$\sum_{k=1}^{N} f(r_k^*, \theta_k^*)\Delta A_k$$

where ΔA_k is the area of the kth polar rectangle.

To find the area of a typical polar rectangle, let (r_k^*, θ_k^*) be the center of the polar rectangle—that is, the point midway between the arcs and rays that form the polar rectangle, as shown in Figure 12.19b. If the circular arcs that bound the polar rectangle are Δr apart, then the arcs are given by

$$r_1 = r_k^* - \tfrac{1}{2}\Delta r_k \qquad \text{and} \qquad r_2 = r_k^* + \tfrac{1}{2}\Delta r_k$$

In Chapter 6 we used the fact that a circular section of radius r and central angle θ has area $\frac{1}{2}r^2\theta$ (see the *Student Mathematics Handbook* for details). Thus, a typical polar rectangle has area

SMH See Section 1.2 of the Student Mathematics Handbook.

$$\Delta A_k = \Big[\underbrace{\tfrac{1}{2}(r_k^* + \tfrac{1}{2}\Delta r_k)^2}_{\text{Radius of outside arc}} - \underbrace{\tfrac{1}{2}(r_k^* - \tfrac{1}{2}\Delta r_k)^2}_{\text{Radius of inside arc}}\Big]\Delta\theta_k = r_k^*\Delta r_k\Delta\theta_k$$

Finally, we compute the given double integral in polar form by taking the limit

$$\iint_D f(r, \theta)\, dA = \lim_{\|P\|\to 0}\sum_{k=1}^{N} f(r_k^*, \theta_k^*)\Delta A_k = \lim_{\|P\|\to 0}\sum_{k=1}^{N} f(r_k^*, \theta_k^*) r_k^*\Delta r_k\Delta\theta_k$$

$$= \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta)\, r\, dr\, d\theta$$

❑

PREVIEW

It can be shown that under reasonable conditions, the change of variable $x = x(u, v)$, $y = y(u, v)$ transforms the integral $\iint f(x, y)\,dA$ into $\iint f(u, v)\,|J(u, v)|\,du\,dv$, where

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

This determinant is known as the **Jacobian** of the transformation. In the case of polar coordinates, we have $x = r\cos\theta$ and $y = r\sin\theta$, so the Jacobian is

$$J(r, \theta) = \begin{vmatrix} \dfrac{\partial}{\partial r}(r\cos\theta) & \dfrac{\partial}{\partial \theta}(r\cos\theta) \\ \dfrac{\partial}{\partial r}(r\sin\theta) & \dfrac{\partial}{\partial \theta}(r\sin\theta) \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

This yields the result of Theorem 12.4:

$$\iint\limits_R f(x, y)\,dA = \iint\limits_D f(r, \theta)\,r\,dr\,d\theta = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

We discuss this more completely in Section 12.6.

If you carefully compare the result in the preview box with the result of Theorem 12.4, you will see that they are not *exactly* the same. The region D in Theorem 12.4 already is in polar coordinates. The more common situation is that we are given f and a region R in rectangular coordinates that we need to change to polar coordinates.

AREA AND VOLUME IN POLAR FORM

SMH See Chapter 5 of the Student *Mathematics Handbook.*

You will need to be familiar with the graphs of many polar-form curves. These can be found in Table 6.2, and also in the *Student Mathematics Handbook*. We now present the example we promised in the Introduction.

EXAMPLE 1 Double integral in polar form

Evaluate $\displaystyle\iint\limits_R (x^2 + y^2 + 1)\,dA$, where D is the region inside the circle $x^2 + y^2 = 4$.

Solution

In this example, the region R is given in rectangular form. We will describe this as a polar region D. Earlier, we observed that in D, for each fixed angle θ between 0 and

(the circle). Thus,

$$\iint_R \underbrace{(x^2+y^2+1)}_{f(x,y)}\,dA = \int_0^{2\pi}\int_0^2 \underbrace{(r^2+1)}_{f(r\cos\theta,\, r\sin\theta)}\ \underbrace{r\,dr\,d\theta}_{dA}$$

$$= \int_0^{2\pi}\int_0^2 (r^3+r)\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}+\frac{r^2}{2}\right]\Bigg|_0^2 d\theta = \int_0^{2\pi} 6\,d\theta = 6\theta\Big|_0^{2\pi} = 12\pi \quad ■$$

EXAMPLE 2 Computing area in polar form using a double integral

Compute the area of the region D bounded above by the line $y = x$ and below by the circle $x^2+y^2-2y=0$.

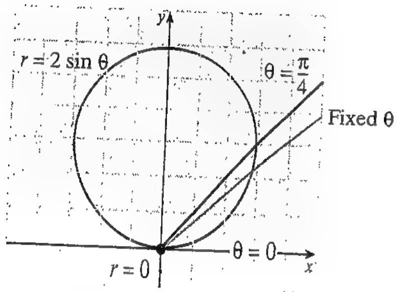


Figure 12.20 The region D

Solution

The circle $x^2+y^2-2y=0$ and the line $y=x$ are shown in Figure 12.20. The polar form for the line $y=x$ is $\theta = \pi/4$ and for the circle is

$$x^2+y^2 = 2y$$
$$r^2 = 2r\sin\theta$$
$$r = 2\sin\theta$$

Thus, the region D is determined as r varies from 0 (the pole) to $2\sin\theta$ (the circle) for θ between 0 and $\pi/4$. The area is given by the integral

$$A = \iint_D dA = \int_0^{\pi/4}\int_0^{2\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\left[\frac{1}{2}r^2\right]\Bigg|_{r=0}^{r=2\sin\theta} d\theta$$

$$= \int_0^{\pi/4}\frac{1}{2}(4\sin^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/4}\frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]\Bigg|_0^{\pi/4} = \frac{\pi-2}{4} \quad ■$$

EXAMPLE 3 Volume in polar form

Use a polar double integral to show that a sphere of radius a has volume $\frac{4}{3}\pi a^3$.

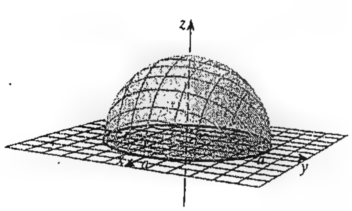


Figure 12.21 The solid hemisphere $z \le \sqrt{a^2-x^2-y^2}$ is bounded above by the sphere and below by the disk $x^2+y^2 \le a^2$.

Solution

We will compute the required volume by doubling the volume of the solid hemisphere $x^2+y^2+z^2 \le a^2$, with $z \ge 0$. This hemisphere (see Figure 12.21) may be regarded as a solid bounded below by the circular disk $x^2+y^2 \le a^2$ and above by the spherical surface.

We need to change the equation of the hemisphere to polar form:

In rectangular form: $\quad z = \sqrt{a^2-x^2-y^2}$

In polar form: $\quad z = \sqrt{a^2-r^2} \quad$ Because $r^2 = x^2+y^2$

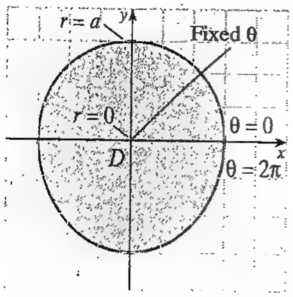


Figure 12.22 The disk D: for θ between 0 and 2π, r varies from 0 to a

Describing the disk D in polar terms, we see that for each fixed θ between 0 and 2π, r varies from the origin to the circle $x^2+y^2=a^2$, which has the polar equation $r=a$, as shown in Figure 12.22. Thus, the volume is given by the integral

$$V = 2\iint_D z\,dA \qquad \text{Rectangular form}$$

$$= 2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\,r\,dr\,d\theta \qquad \boxed{\text{Let } u=a^2-r^2;\ du=-2r\,dr}$$

$$= 2\int_0^{2\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]\Bigg|_{r=0}^{r=a} d\theta$$

$$= -\frac{2}{3}\int_0^{2\pi}[(a^2-a^2)^{3/2}-(a^2-0)^{3/2}]\,d\theta$$

$$= \frac{2}{3}\int_0^{2\pi} a^3\,d\theta = \frac{2}{3}a^3\theta\Big|_0^{2\pi} = \frac{4}{3}\pi a^3$$

EXAMPLE 4 Region of integration between two polar curves

Evaluate $\iint_D \frac{1}{x}\,dA$; where D is the region that lies inside the circle $r=3\cos\theta$ and outside the cardioid $r=1+\cos\theta$.

Solution

Begin by sketching the given curves, as shown in Figure 12.23. Next, find the points of intersection:

$$3\cos\theta = 1+\cos\theta$$
$$2\cos\theta = 1$$
$$\cos\theta = \tfrac{1}{2}$$
$$\theta = -\tfrac{\pi}{3},\ \tfrac{\pi}{3}$$

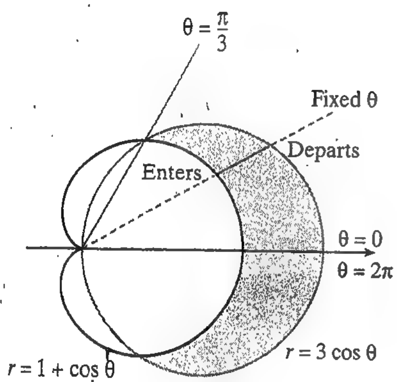


Figure 12.23 Region D

Notice that D is the region such that, for each fixed angle θ between $-\pi/3$ and $\pi/3$, r varies from $1+\cos\theta$ (the cardioid) to $3\cos\theta$ (the circle). This gives us the limits of integration, and in polar form, the integrand becomes

$$\underbrace{\frac{1}{x}}_{\text{Rectangular form}} = \underbrace{\frac{1}{r\cos\theta}}_{\text{Polar form}}$$

Thus,

$$\iint_D \frac{1}{x}\,dA = \iint_D \frac{1}{r\cos\theta}\,r\,dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3}\int_{1+\cos\theta}^{3\cos\theta}\frac{1}{\cos\theta}\,dr\,d\theta$$

$$= 2\int_0^{\pi/3}\left[\frac{r}{\cos\theta}\right]\Bigg|_{r=1+\cos\theta}^{r=3\cos\theta} d\theta \qquad \text{By symmetry}$$

$$= 2\int_0^{\pi/3}\frac{1}{\cos\theta}[3\cos\theta-(1+\cos\theta)]\,d\theta$$

$$= 2\int_0^{\pi/3}(2-\sec\theta)\,d\theta$$

$$= 2[2\theta-\ln|\sec\theta+\tan\theta|]\big|_0^{\pi/3}$$

$$= \frac{4\pi}{3}-2\ln(2+\sqrt{3})$$

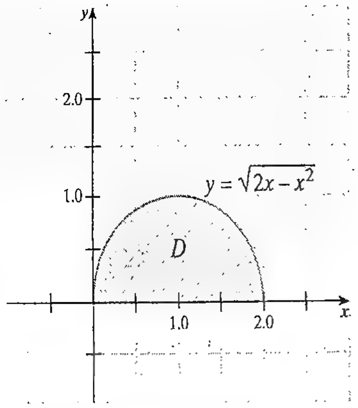


Figure 12.24 Region of integration in Example 5

EXAMPLE 5 Converting an integral to polar form

Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\,dy\,dx$ by converting to polar coordinates.

Solution

The region of integration D is the set of all points (x, y) such that for each point x in the interval $[0, 2]$, y varies from $y = 0$ to $y = \sqrt{2x - x^2}$. Note that

$$\begin{aligned} y &= \sqrt{2x - x^2} \\ y^2 &= 2x - x^2 \\ x^2 - 2x + y^2 &= 0 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

This is the equation of a circle of radius 1 centered at $(1, 0)$. Thus, $y = \sqrt{2x - x^2}$ is the semicircle shown in Figure 12.24. The top boundary of this region is the semicircle

$$\begin{aligned} y &= \sqrt{2x - x^2} \\ y^2 &= 2x - x^2 \\ x^2 + y^2 &= 2x \end{aligned}$$

which has the polar form

$$\begin{aligned} r^2 &= 2r\cos\theta \\ r &= 2\cos\theta \end{aligned}$$

In polar coordinates, D is the region in which r varies from 0 to $2\cos\theta$ for θ between 0 and $\frac{\pi}{2}$. Thus, the integral is

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\,dy\,dx &= \iint_D y\sqrt{x^2+y^2}\,dA \\ &= \int_0^{\pi/2} \int_0^{2\cos\theta} (r\sin\theta)\,r\,(r\,dr\,d\theta) = \int_0^{\pi/2} \sin\theta \left[\frac{r^4}{4}\right]\Bigg|_0^{2\cos\theta} d\theta \\ &= \frac{1}{4}\int_0^{\pi/2} 16\cos^4\theta\sin\theta\,d\theta = 4\left[-\frac{1}{5}\cos^5\theta\right]\Bigg|_0^{\pi/2} = \frac{4}{5} \end{aligned}$$

■

12.3 PROBLEM SET (pg 206 #HW copy① - pg 16 #HW copy②)

Guideline 3 1-23

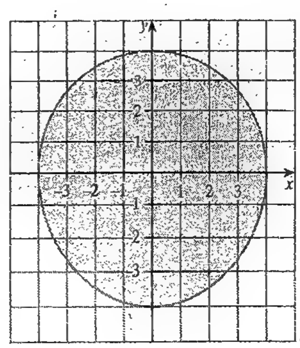
A *In Problems 1–2, sketch the region D and then evaluate the double integral* $\iint_D f(r,\theta)\,dr\,d\theta$.

1. $\int_0^{\pi/2} \int_1^3 re^{-r^2}\,dr\,d\theta$

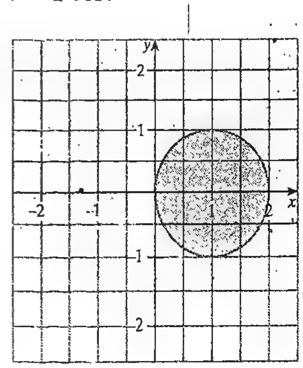
2. $\int_0^{\pi} \int_0^4 r^2 \sin^2\theta\,dr\,d\theta$

Use a double integral to find the area of the shaded region in Problems 3–12.

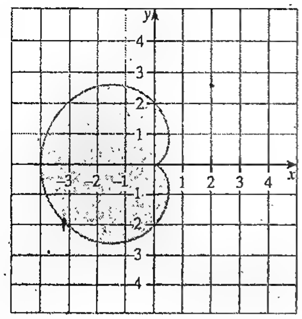
3. $r = 4$



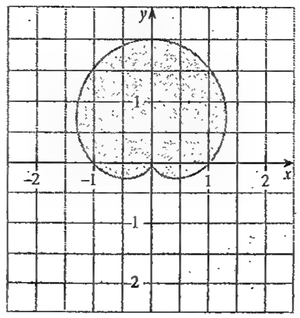
4. $r = 2\cos\theta$



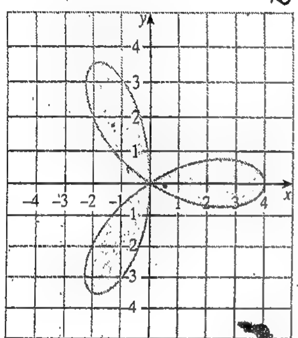
5. $r = 2(1 - \cos\theta)$



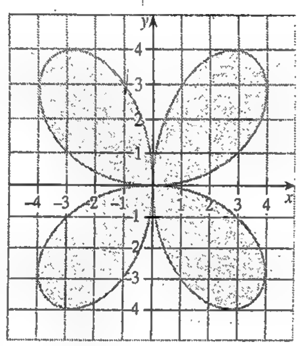
6. $r = 1 + \sin\theta$



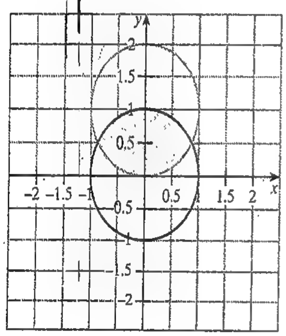
7. $r = 4\cos 3\theta$



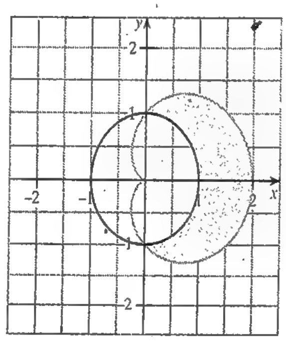
8. $r = 5\sin 2\theta$



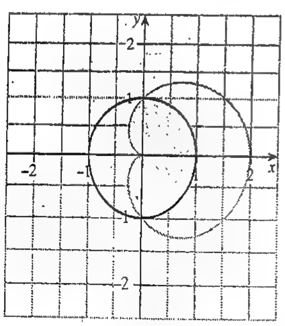
9. $r = 1$ and $r = 2\sin\theta$
Hint: Consider $0 < \theta < \frac{\pi}{6}$ and $\frac{\pi}{6} < \theta < \frac{\pi}{2}$ separately.



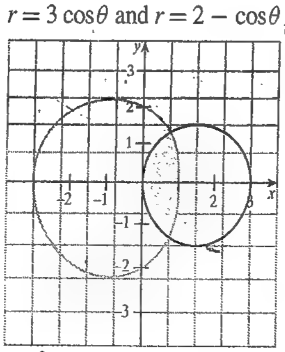
10. $r = 1$ and $r = 1 + \cos\theta$



11. $r = 1$ and $r = 1 + \cos\theta$



12. $r = 3\cos\theta$ and $r = 2 - \cos\theta$



13. $f(x, y) = y^2$

14. $f(x, y) = \dfrac{1}{a^2 + x^2 + y^2}$

15. $f(x, y) = e^{-(x^2 + y^2)}$

Use polar coordinates in Problems 16–17 to evaluate the given double integral.

16. $\displaystyle\iint_D y\,dA$, where D is the disk $x^2 + y^2 \le 4$

17. $\displaystyle\iint_D e^{x^2+y^2}\,dA$, where D is the region inside the circle $x^2 + y^2 = 9$

In Problems 18—20, evaluate the given integral by converting to polar coordinates.

18. $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} x\,dy\,dx$

19. $\displaystyle\int_0^2 \int_y^{\sqrt{8-y^2}} \frac{1}{\sqrt{1 + x^2 + y^2}}\,dx\,dy$

20. $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} \frac{1}{\sqrt{9 - x^2 - y^2}}\,dx\,dy$

B 21. Use polar coordinates to evaluate $\iint_D xy\,dA$, where D is the intersection of the circular disks $r \le 4\cos\theta$ and $r \le 4\sin\theta$. Sketch the region of integration.

C 22. **Exploration Problem** If we evaluate the integral $\iint_R r^2\,dr\,d\theta$, where R is the region in the xy-plane bounded by $r = 2\cos\theta$, we obtain

$$\int_0^{\pi}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_0^{\pi} \frac{r^3}{3}\bigg|_0^{2\cos\theta} d\theta$$
$$= \frac{8}{3}\int_0^{\pi} \cos^3\theta\,d\theta = \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]\bigg|_0^{\pi} = 0$$

Alternatively, we can set up the integral as

$$\iint_R r^2\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta$$
$$+ \int_{\pi/2}^{\pi}\int_0^{2\cos\theta} r^2(-dr)\,d\theta = \frac{32}{9}$$

Both of these answers cannot be correct. Which procedure (if either) is correct and why?

23. **HISTORICAL QUEST** Newton and Leibniz have been credited with the discovery of calculus, but much of its development was due to the mathematicians Pierre-Simon Laplace, Lagrange (Historical Quest 55, Section 11.8), and Gauss (Historical Quest 1, Supplementary Problems to Chapter 5). These three great mathematicians of calculus were contrasted by W.W. Rouse Ball:

PIERRE-SIMON LAPLACE 1749–1827

> The great masters of modern analysis are Lagrange, Laplace, and Gauss, who were contemporaries. It is interesting to note the marked contrast in their styles. Lagrange is perfect both in form and matter, he is careful to explain his procedure, and though his arguments are general they are easy to follow. Laplace on the other hand explains nothing, is indifferent to style, and, if satisfied that his results are correct, is content to leave them either with no proof or with a faulty one.

Gauss is exact and elegant as Lagrange, but even more difficult to follow than Laplace, for he removes every trace of the analysis by which he reached his results, and strives to give a proof which while rigorous will be as concise and synthetical as possible.*

Pierre-Simon Laplace has been called the Newton of France. He taught Napoleon Bonaparte, was appointed for a time as Minister of Interior, and was at times granted favors from his powerful friend. Today, Laplace is best known as the major contributor to probability, taking it from gambling to a true branch of mathematics. He was one of the earliest to evaluate the improper integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

which plays an important role in the theory of probability. ■

Show that $I = \sqrt{\pi}$. *Hint*: Note that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \cdot \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

where the integral on the right is over the entire xy-plane, described in polar terms by $0 \le r < \infty$ and $0 \le \theta \le 2\pi$.

A Short Account of the History of Mathematics, as quoted in *Mathematical Circles Adieu*, by Howard Eves (Boston: Prindle, Weber & Schmidt, Inc., 1977).

12.4 Triple Integrals

IN THIS SECTION definition of the triple integral, iterated integration, volume by triple integrals

DEFINITION OF THE TRIPLE INTEGRAL

A double integral $\iint\limits_D f(x, y)\,dA$ is evaluated over a closed, bounded region in the plane, and in essentially the same way, a **triple integral** $\iiint\limits_D f(x, y, z)\,dV$ is evaluated over a closed, bounded solid region D in $\mathbb{R}^3$. Suppose $f(x, y, z)$ is defined on a closed region D, which in turn is contained in a "box" B in space. Partition B into a finite number of smaller boxes using planes parallel to the coordinate planes, as shown in Figure 12.25.

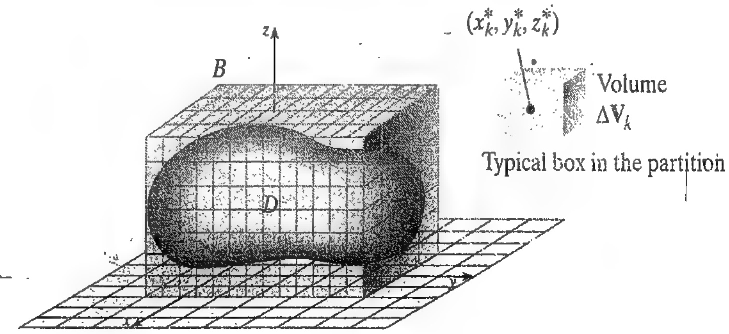


Figure 12.25 The box B contains D and is subdivided into smaller boxes.

We exclude from consideration any boxes with points outside S. Let ΔV_1, ΔV_2, …, ΔV_n denote the volumes of the boxes that remain, and define the norm $\|P\|$ of the partition to be the length of the longest diagonal of any box in the partition. Next, choose a representative point (x_k^*, y_k^*, z_k^*) from each box in the partition and form the Riemann sum

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

If we repeat the process with more subdivisions, so that the norm approaches zero, we are led to the following definition.

Triple Integral

If f is a function defined over a closed bounded solid region D, then the **triple integral of f over D** is defined to be the limit

$$\iiint_D f(x, y, z)\,dV = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

provided this limit exists.

A surface is said to be **piecewise smooth** if it is made up of a finite number of smooth surfaces. It can be shown that the triple integral exists if $f(x, y, z)$ is continuous on D and the surface of D is piecewise smooth. It can also be shown that triple integrals have the following properties, which are analogous to those of double integrals listed in Theorem 12.1. In each case, assume the indicated integrals exist.

Linearity rule For constants a and b

$$\iiint_D [af(x, y, z) + bg(x, y, z)]\,dV$$

$$= a\iiint_D f(x, y, z)\,dV + b\iiint_D g(x, y, z)\,dV$$

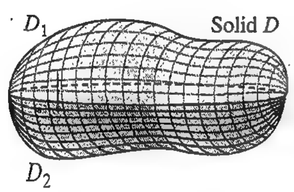


Figure 12.26 Dividing the solid D into two solid subregions

Dominance rule If $f(x, y, z) \geq g(x, y, z)$ on D, then

$$\iiint_D f(x, y, z)\,dV \geq \iiint_D g(x, y, z)\,dV$$

Subdivision rule If the solid region of integration D can be subdivided into two solid subregions D_1 and D_2 (see Figure 12.26), then

$$\iiint_D f(x, y, z)\,dV = \iiint_{D_1} f(x, y, z)\,dV + \iiint_{D_2} f(x, y, z)\,dV$$

ITERATED INTEGRATION

As with double integrals, we evaluate triple integrals by iterated integration. However, setting up the limits of integration in an iterated triple integral is often difficult, especially if the solid region of integration, D, is hard to visualize. The relatively simple case where B is a rectangular solid (box) may be handled by applying the following theorem.

THEOREM 12.5 **Fubini's theorem over a parallelepiped in space**

If $f(x, y, z)$ is continuous over a rectangular box B: $a \leq x \leq b$, $c \leq y \leq d$, $r \leq z \leq s$; then the triple integral may be evaluated by the iterated integral

$$\iiint_B f(x, y, z)\,dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\,dx\,dy\,dz$$

The iterated integration can be performed in any order, with appropriate adjustments to the limits of integration:

$$\begin{array}{ccc} dx\,dy\,dz & dx\,dz\,dy & dz\,dx\,dy \\ dy\,dx\,dz & dy\,dz\,dx & dz\,dy\,dx \end{array}$$

Proof The proof, which is similar to the two-dimensional case, can be found in an advanced calculus course. ❑

EXAMPLE 1 Evaluating a triple integral using Fubini's theorem

Evaluate $\iiint\limits_B z^2 y e^x \, dV$, where B is the box given by

$$0 \le x \le 1, \quad 1 \le y \le 2, \quad -1 \le z \le 1$$

This box is shown in Figure 12.27.

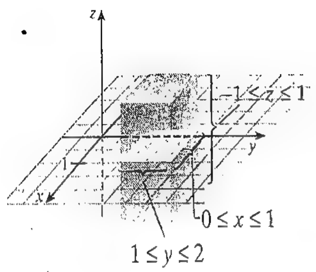


Figure 12.27 Region B

Solution

We will evaluate the integral in the order $dx\,dy\,dz$.

$$\iiint\limits_B f(x, y, z)\,dV = \int_{-1}^{1}\int_{1}^{2}\underbrace{\int_{0}^{1} z^2 y e^x \, dx}_{\text{Treat } y \text{ and } z \text{ as constants.}}\, dy\, dz$$

$$= \int_{-1}^{1}\int_{1}^{2} z^2 y \left[e^x\right]\Big|_{x=0}^{x=1} dy\, dz = \int_{-1}^{1}\underbrace{\int_{1}^{2} z^2 y\,[e-1]\, dy}_{\text{Treat } z \text{ as a constants.}}\, dz$$

$$= (e-1)\int_{-1}^{1} z^2\left[\frac{y^2}{2}\right]\Bigg|_{y=1}^{y=2} dz = (e-1)\int_{-1}^{1} z^2\left[\frac{2^2}{2} - \frac{1^2}{2}\right] dz$$

$$= \frac{3}{2}(e-1)\int_{-1}^{1} z^2\, dz = \frac{3}{2}(e-1)\,\frac{z^3}{3}\Bigg|_{z=-1}^{z=1}$$

$$= \frac{3}{2}(e-1)\left[\frac{1^3}{3} - \frac{(-1)^3}{3}\right] = e - 1$$

As an exercise, verify that the same result is obtained by using any other order of integration—for example, $dz\,dy\,dx$. ■

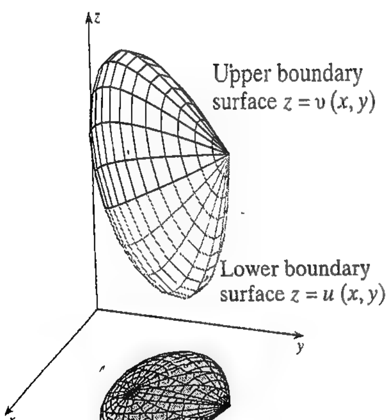


Figure 12.28 A z-simple solid D

Next, we will see how triple integrals can be evaluated over solid regions that are not rectangular boxes. We will assume the solid region of integration D is *z-simple* in the sense that it has a lower bounding surface $z = u(x, y)$ and an upper bounding surface $z = v(x, y)$, and that it projects onto a region A in the xy-plane that is of either type I or type II. Such a solid is shown in Figure 12.28.

The region D can be described as the set of all points (x, y, z) such that $u(x, y) \le z \le v(x, y)$ for all (x, y) in A. Then the triple integral of $f(x, y, z)$ over the solid region D can be expressed as an iterated integral with inner limits of integration (with respect to z) $u(x, y)$ and $v(x, y)$. We summarize in the following theorem.

THEOREM 12.6 **Triple integral over a z-simple solid region**

Suppose D is a solid region bounded below by the surface $z = u(x, y)$ and above by $z = v(x, y)$ that projects onto the region A in the xy-plane. If A is of either type I or

type II, then the integral of the continuous function $f(x, y, z)$ over D is

$$\iiint\limits_D f(x, y, z)\,dV = \iint\limits_A \left(\int_{u(x,y)}^{v(x,y)} f(x, y, z)\,dz\right) dA$$

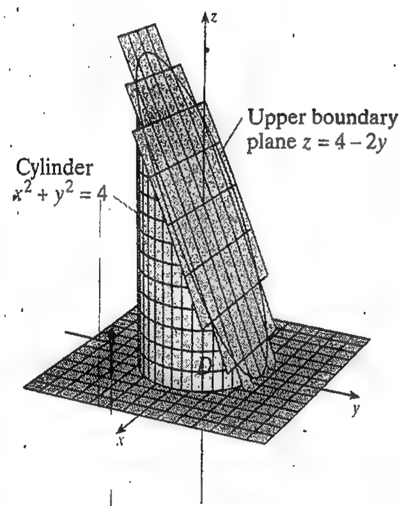
Proof Even though a proof is beyond the scope of this text, we note that if A is vertically simple (type I), then for each fixed x in an interval $[a, b]$, y varies from $g_1(x)$ to $g_2(x)$, and the triple integral of $f(x, y, z)$ over D can be expressed as

$$\iiint\limits_D f(x, y, z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u(x,y)}^{v(x,y)} f(x, y, z)\,dz\,dy\,dx$$

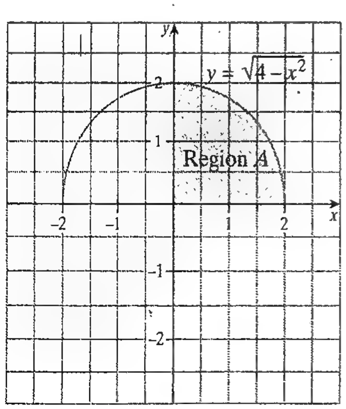
Likewise, if A is horizontally simple (type II), then for each fixed y in an interval $[c, d]$, x varies from $h_1(y)$ to $h_2(y)$, and

$$\iiint\limits_D f(x, y, z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u(x,y)}^{v(x,y)} f(x, y, z)\,dz\,dx\,dy$$ ❑

We illustrate this procedure in the following example.



a. The solid D



b. The region A

Figure 12.29 The solid D and its projection A in the xy-plane

EXAMPLE 2 Evaluating a triple integral over a general region

Evaluate $\iiint\limits_D x\,dV$, where D is the solid in the first octant bounded by the cylinder $x^2 + y^2 = 4$ and the plane $2y + z = 4$.

Solution

The solid is shown in Figure 12.29. The upper boundary surface of D is the plane $z = 4 - 2y$, and the lower boundary surface is the xy-plane, $z = 0$. The projection A of the solid on the xy-plane is the quarter disk $x^2 + y^2 \leq 4$ with $x \geq 0$, $y \geq 0$ (because D lies in the first octant). This projection may be described in type I form as the set of all (x, y) such that for each fixed x between 0 and 2, y varies from 0 to $\sqrt{4 - x^2}$. Thus, we have

$$\begin{aligned}
\iiint\limits_D x\,dV &= \iint\limits_A \int_0^{4-2y} x\,dz\,dA = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-2y} x\,dz\,dy\,dx \\
&= \int_0^2 \int_0^{\sqrt{4-x^2}} x[(4-2y) - 0]\,dy\,dx = \int_0^2 \int_0^{\sqrt{4-x^2}} (4x - 2xy)\,dy\,dx \\
&= \int_0^2 \left[4xy - xy^2\right]\Big|_{y=0}^{y=\sqrt{4-x^2}} dx = \int_0^2 \left[4x\sqrt{4-x^2} - x(4 - x^2)\right] dx \\
&= \left[-\frac{4}{3}(4 - x^2)^{3/2} - 2x^2 + \frac{1}{4}x^4\right]\Big|_0^2 = \left[0 - 8 + 4 + \frac{32}{3} + 0 - 0\right] = \frac{20}{3}
\end{aligned}$$ ■

Just as a double integral can be interpreted as the area of the region of integration, a triple integral may be interpreted as the **volume** of a solid. That is, if V is the volume of the solid region D, then

$$\boxed{V = \iiint_D dV}$$

EXAMPLE 3 Volume of a tetrahedron

Find the volume of the tetrahedron T bounded by the plane $2x + y + 3z = 6$ and the coordinate planes $x = 0$, $y = 0$, and $z = 0$.

Solution

The tetrahedron T is shown in Figure 12.30a. The upper surface of T is the plane $z = \frac{1}{3}(6 - 2x - y)$ and its lower surface is $z = 0$. Note that T projects onto a triangle, A, in the xy-plane, as shown in Figure 12.30b. Described in type I form, the triangle A is the set of all (x, y) such that for each fixed x between 0 and 3, y varies from 0 to $6 - 2x$. Thus,

$$V = \iiint_T dV = \iint_A \int_0^{\frac{1}{3}(6-2x-y)} dz\, dA$$

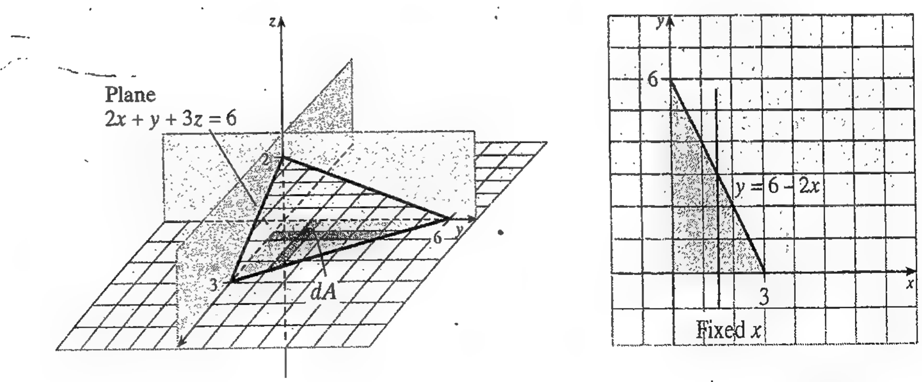
$$= \int_0^3 \int_0^{6-2x} \int_0^{\frac{1}{3}(6-2x-y)} dz\, dy\, dx$$

$$= \int_0^3 \int_0^{6-2x} \left[\tfrac{1}{3}(6 - 2x - y) - 0\right] dy\, dx$$

$$= \int_0^3 \left[2y - \tfrac{2}{3}xy - \tfrac{1}{6}y^2\right]\Big|_{y=0}^{y=6-2x} dx$$

$$= \int_0^3 \left[2(6 - 2x) - \tfrac{2}{3}x(6 - 2x) - \tfrac{1}{6}(6 - 2x)^2 - 0\right] dx$$

$$= \int_0^3 \tfrac{1}{6}[36 - 24x + 4x^2]\, dx = 6$$



a. The tetrahedron bounded by the plane $2x + y + 3z = 6$ and the positive coordinate planes.

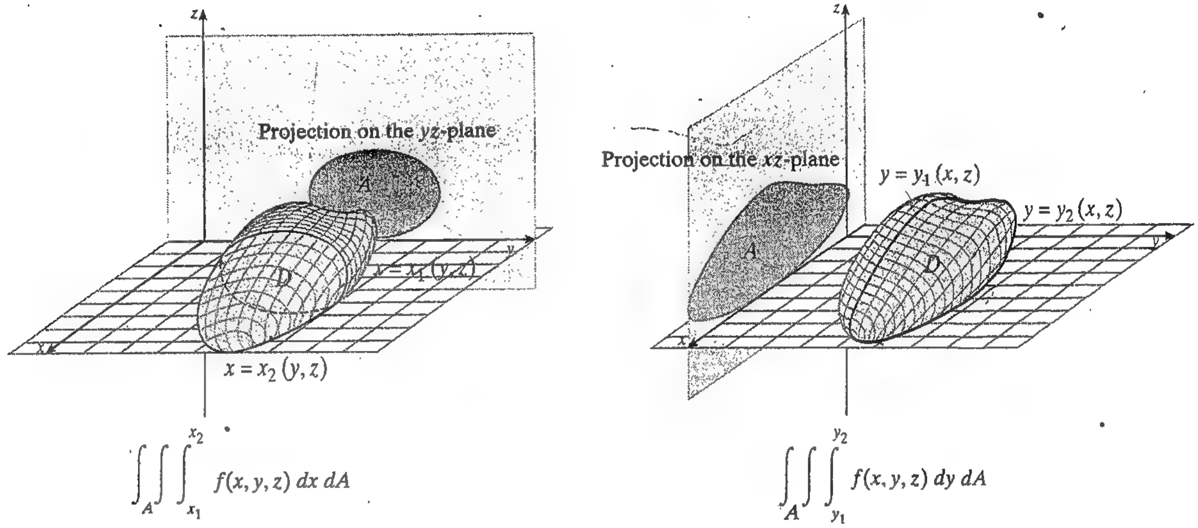
b. The region projected onto the xy-plane is a triangle.

Figure 12.30 Volume of a tetrahedron

The volume of the tetrahedron is 6 cubic units. ■

Sometimes it is easier to evaluate a triple integral by integrating first with respect to x or y instead of z. For instance, if the solid region of integration D is bounded by $x = x_1(y, z)$ and $x = x_2(y, z)$, and the boundary surfaces project onto a region A in the yz-plane, as shown in Figure 12.31a, then

$$\iiint\limits_D f(x, y, z)\, dV = \iint\limits_A \int_{x_1}^{x_2} f(x, y, z)\, dx\, dA$$



a. A solid D with "front" surface $x = x_2(y, z)$ and "back" surface $x = x_1(y, z)$

b. A solid D with "side" surfaces $y = y_2(x, z)$ and $y = y_1(x, z)$

Figure 12.31 Iterated integration with respect to x or y first

On the other hand, if the solid region of integration D is bounded by the surfaces $y = y_1(x, z)$ and $y = y_2(x, z)$, and the boundary surfaces project onto a region A in the xz-plane as shown in Figure 12.31b, then

$$\iiint\limits_D f(x, y, z)\, dV = \iint\limits_A \int_{y_1}^{y_2} f(x, y, z)\, dy\, dA$$

As an illustration, we will now rework Example 3 by projecting the tetrahedron T onto the yz-plane.

EXAMPLE 4 Volume of a tetrahedron by changing the order of integration

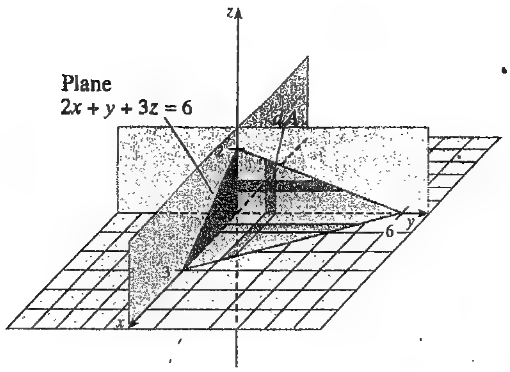
Find the volume of the tetrahedron T bounded by the coordinate planes and the plane $2x + y + 3z = 6$ in the first octant by projecting onto the yz-plane.

Solution

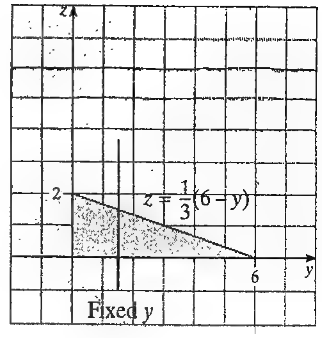
Note that T is bounded by the yz-plane and the plane $2x + y + 3z = 6$, which we express as $x = \frac{1}{2}(6 - y - 3z)$. (See Figure 12.32a.) The volume is given by

$$V = \iiint\limits_T dV = \iint\limits_D \int_0^{\frac{1}{2}(6-y-3z)} dx\, dA$$

where D is the projection in the yz-plane. This projection is the triangle bounded by the lines $z = 0$, $y = 0$, and $z = \frac{1}{3}(6 - y)$, as shown in Figure 12.32b.



a. The tetrahedron bounded by the plane $2x + y + 3z = 6$ and the positive coordinate planes.



b. The region projected onto the yz-plane is a triangle.

Figure 12.32 Volume of a tetrahedron; alternative projection

Thus, for each fixed y between 0 and 6, z varies from 0 to $\frac{1}{3}(6 - y)$, and we have

$$V = \int_0^6 \int_0^{\frac{1}{3}(6-y)} \int_0^{\frac{1}{2}(6-y-3z)} dx\,dz\,dy = \int_0^6 \int_0^{\frac{1}{3}(6-y)} \tfrac{1}{2}(6 - y - 3z)\,dz\,dy$$

$$= \int_0^6 \left[3z - \tfrac{1}{2}yz - \tfrac{3}{4}z^2\right]\Big|_{z=0}^{z=\frac{1}{3}(6-y)} dy$$

$$= \int_0^6 \left[(6 - y) - \tfrac{1}{6}y(6 - y) - \tfrac{1}{12}(6 - y)^2 - 0\right] dy = 6$$

This is the same result we obtained in Example 3 by projecting onto the xy-plane. ■

EXAMPLE 5 Setting up a triple integral to find a volume

Set up (but do not evaluate) a triple integral for the volume of the solid D that is bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $y + z = 2$. The projection on the xy-plane is shown as a shadow in Figure 12.33.

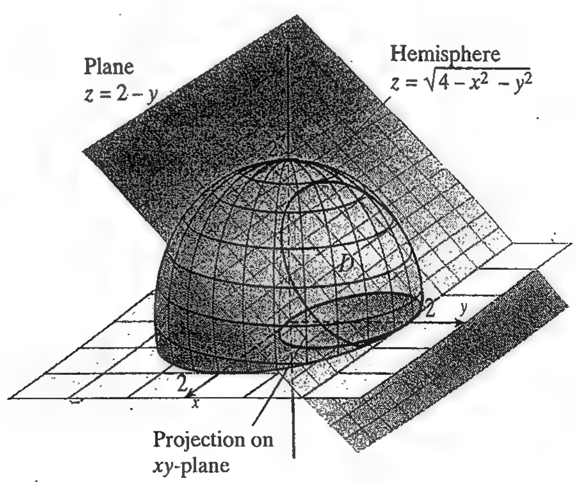


Figure 12.33 Region bounded above by a sphere and below by a plane

Solution

First, note that the intersection of the plane and the sphere occurs above the xy-plane (where $z \geq 0$), so the sphere can be represented by the equation

$$z = \sqrt{4 - x^2 - y^2}$$

(the upper hemisphere). To find the limits of integration for x and y we consider the projection of D onto the xy-plane. To this end, consider the intersection of the hemisphere and the plane $z = 2 - y$:

$$\sqrt{4 - x^2 - y^2} = 2 - y$$
$$4 - x^2 - y^2 = 4 - 4y + y^2$$
$$x^2 + 2y^2 - 4y = 0$$
$$x^2 + 2(y - 1)^2 = 2$$

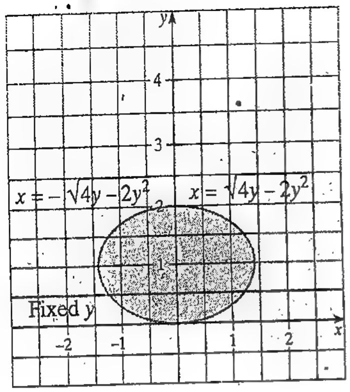


Figure 12.34 The projection of D onto the xy-plane

Although this intersection occurs in $\mathbb{R}^3$, its equation does not contain z. Therefore, the equation serves as a projection on the xy-plane, where $z = 0$. This is an ellipse centered at (0, 1), as shown in Figure 12.34.

We consider this as a type II region, which means we will integrate first with respect to x, then with respect to y. Because $x^2 + 2(y - 1)^2 = 2$ we see that for fixed y between 0 and 2, x varies from $-\sqrt{2 - 2(y-1)^2} = -\sqrt{4y - 2y^2}$ to $\sqrt{4y - 2y^2}$. However, using symmetry, we see the required volume V is twice the integral as x varies from 0 to $\sqrt{4y - 2y^2}$. This leads us to evaluate V by the following triple integral.

$$V = 2\int_0^2 \int_0^{\sqrt{4y-2y^2}} \int_{2-y}^{\sqrt{4-x^2-y^2}} dz\,dx\,dy$$ ■

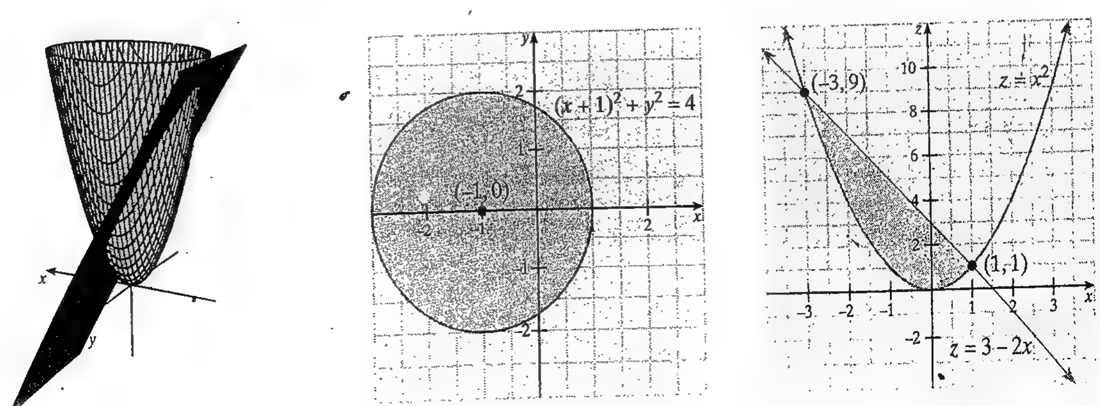
EXAMPLE 6 Choosing an order of integration to compute volume

Find the volume of the solid D bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $2x + z = 3$.

Solution

The graph of the solid D is shown in Figure 12.35a. The projection of the solid onto the xy-plane is the graph of the equation

$x^2 + y^2 = 3 - 2x$	Substitute the second equation into the first.
$(x + 1)^2 + y^2 = 4$	Complete the square for x.



a. The solid D bounded by $z = x^2 + y^2$ and $2x + z = 3$

b. The projection of D on the xy-plane

c. The projection of D on the xz-plane

Figure 12.35 The solid D bounded by $z = x^2 + y^2$ and $2x + z = 3$ and projections on two coordinate planes

This is a circle of radius 2 centered at $(-1, 0)$, as shown in Figure 12.35b. Proceeding as in Example 5, we find that the volume is given by the integral

$$V = 2\int_{-3}^{1}\int_{0}^{\sqrt{3-2x-x^2}}\int_{x^2+y^2}^{3-2x} dz\,dy\,dx$$

which is not especially easy to integrate (try it!).

However, if we project the solid D onto the xz-plane, the projection A is the region bounded by the line $2x + z = 3$ and the parabola $z = x^2$ (take $y = 0$ in the equation $z = x^2 + y^2$), which intersect at the points $(1, 1)$ and $(-3, 9)$, as shown in Figure 12.35c. The solid D is symmetric with respect to the xz-plane, so we can integrate with respect to y from $y = 0$ to $y = \sqrt{z - x^2}$ and double the result. We can then describe A as the following type I region.

A: set of all (x, z) such that for each fixed x in the interval $-3 \le x \le 1$, z varies from $z = x^2$ to $z = 3 - 2x$

The volume of D is given by the integral

$$\begin{aligned}
V &= 2\int_{-3}^{1}\int_{x^2}^{3-2x}\int_{0}^{\sqrt{z-x^2}} dy\,dz\,dx \\
&= 2\int_{-3}^{1}\int_{x^2}^{3-2x}\sqrt{z-x^2}\,dz\,dx \\
&= 2\int_{-3}^{1}\frac{2}{3}(z-x^2)^{3/2}\Big|_{x^2}^{3-2x} dx \\
&= \frac{4}{3}\int_{-3}^{1}(3-2x-x^2)^{3/2}\,dx \\
&= \frac{4}{3}\int_{-3}^{1}[4-(x+1)^2]^{3/2}\,dx && \text{Complete the square.} \\
& && \boxed{\text{Let } x+1 = 2\sin\theta;\ dx = 2\cos\theta\, d\theta} \\
&= \frac{4}{3}\int_{-\pi/2}^{\pi/2} 8\cos^3\theta\,(2\cos\theta\, d\theta) \\
&= \frac{128}{3}\int_{0}^{\pi/2}\cos^4\theta\, d\theta && \text{Symmetry} \\
&= \frac{128}{3}\left(\frac{3\pi}{16}\right) && \text{Formula 320} \\
&= 8\pi
\end{aligned}$$

■

12.4 PROBLEM SET (Pg32 – Pg66)

1. **WHAT DOES THIS SAY?** State Fubini's theorem for a continuous function over a parallelepiped in $\mathbb{R}^3$.

Compute the iterated triple integrals in Problems 2–9.

2. $\int_1^4 \int_{-2}^3 \int_2^5 dx\,dy\,dz$

3. $\int_1^2 \int_0^1 \int_{-1}^2 8x^2yz^3\,dx\,dy\,dz$

4. $\int_0^2 \int_0^x \int_0^{x+y} xyz\,dz\,dy\,dx$

5. $\int_{-1}^2 \int_0^1 \int_1^4 yz\cos xy\,dz\,dx\,dy$

6. $\int_0^\pi \int_0^1 \int_0^1 x^2y\cos xyz\,dz\,dy\,dx$

7. $\int_0^1 \int_0^y \int_0^{\ln y} e^{z+2x}\,dz\,dx\,dy$

8. $\int_1^4 \int_{-1}^{2z} \int_0^{\sqrt{3}x} \frac{x-y}{x^2+y^2}\,dy\,dx\,dz$

9. $\int_0^1 \int_{x-1}^{x^2} \int_{-x}^{y} (x+y)\,dz\,dy\,dx$

Evaluate the triple integrals in Problems 10–14.

10. $\iiint_D (x^2y + y^2z)\,dV$, where D is the box $1 \le x \le 3, -1 \le y \le 1, 2 \le z \le 4$

11. $\iiint_D (xy + 2yz)\,dV$, where D is the box $2 \le x \le 4, 1 \le y \le 3, -2 \le z \le 4$

12. $\iiint_D xyz\,dV$, where D is the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

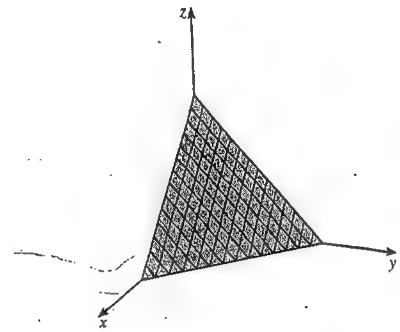
13. $\iiint_D xyz\,dV$, where D is the region given by $x^2 + y^2 + z^2 \le 1$, $y \ge 0, z \ge 0$

14. $\iiint_D e^z\,dV$, where D is the region described by the inequalities $0 \le x \le 1, 0 \le y \le x$, and $0 \le z \le x + y$

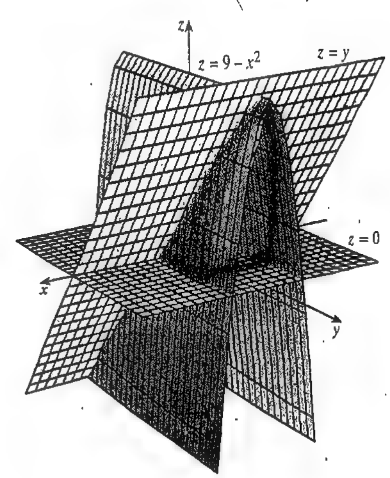
Guidelines 1–26, 28, 29, 31

Find the volume V of the solids bounded by the graphs of the equations given in Problems 15–22 by using triple integration.

15. $x + y + z = 1$ and the coordinate planes



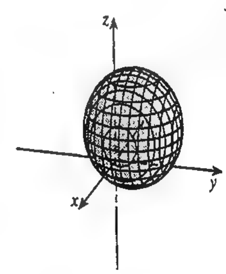
16. $z = 9 - x^2, z = 0, y = z$



Pg 102 HW②

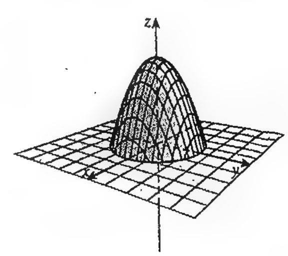
17. $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 1$

Pg59 HW②

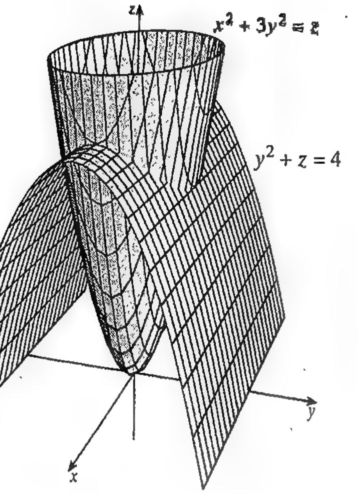


using Jacobian down on Pg59

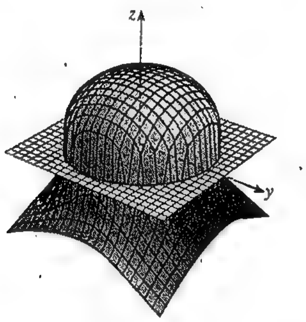
18. $z = 4 - 4x^2 - 4y^2, z = 0$



19. $x^2 + 3y^2 = z$ and the cylinder $y^2 + z = 4$



20. $x^2 + y^2 + z^3 = 9, z = 0$



21. The solid bounded above by the paraboloid $z = 6 - x^2 - y^2$ and below by $z = 2x^2 + y^2$

22. The solid region common to the cylinders $x^2 + z^2 = 1$ and $x^2 + y^2 = 1$

For each given iterated integral there are five other equivalent iterated integrals. Find the one with the requested order in Problems 23–25.

23. $\int_0^1 \int_0^x \int_0^y f(x, y, z)\, dz\, dy\, dx$; change the order to $dz\, dx\, dy$.

24. $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} f(x, y, z)\, dz\, dy\, dx$; change the order to $dx\, dy\, dz$.

25. $\int_0^{1/2} \int_0^{1-4x^2} \int_0^{1-2x} f(x, y, z)\, dz\, dy\, dx$; change the order $dy\, dx\, dz$.

26. Find the volume of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$.

27. A wedge is cut from a right-circular cylinder of radius R by a plane perpendicular to the axis of the cylinder and a second plane that meets the first on the axis at an angle of θ degrees, as shown in Figure 12.36.

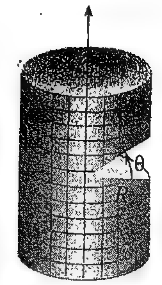


Figure 12.36 Cutting a wedge from a cylinder

Set up and evaluate a triple integral for the volume of the wedge.

28. Find the volume of the solid region in the first octant that is bounded by the planes $z = 8 + 2x + y$ and $y = 3 - 2x$.

29. Use triple integration to find the volume of a sphere.

30. Use triple integration to find the volume of a right pyramid with height H and a square base of side S.

31. Use triple integration to find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(assume $a > 0, b > 0, c > 0$).

32. **Counterexample Problem** Let B be the box defined by $a \le x \le b, c \le y \le d, r \le z \le s$. Is it true that

$$\iiint_B f(x)g(y)h(z)\, dV = \left[\int_a^b f(x)\, dx\right]\left[\int_c^d g(y)\, dy\right]\left[\int_r^s h(z)\, dz\right],$$

if f, g, and h are continuous? Either show that this equation is generally true or find a counterexample.

33. One of the following integrals has the value 0. Which is it and why?

A. $\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} (x + z^2)\, dz\, dx\, dy$

B. $\int_0^1 \int_x^{2-x^2} \int_{-3}^{3} z^2 \sin xz\, dz\, dy\, dx$

Higher-dimensional multiple integrals can be defined and evaluated in essentially the same way as double integrals and triple integrals. Evaluate the given multiple integrals in Problem 34.

34. $\iiiint_H e^{x-2y+z+w}\, dw\, dz\, dy\, dx,$

where H is the four-dimensional region bounded by the hyperplane $x + y + z + w = 4$ and the coordinate spaces $x = 0$, $y = 0$, $z = 0$, and $w = 0$ in the first hyperoctant (where $x \ge 0$, $y \ge 0$, $z \ge 0$, $w \ge 0$)

12.5 Cylindrical and Spherical Coordinates

IN THIS SECTION cylindrical coordinates, integration with cylindrical coordinates, spherical coordinates, integration with spherical coordinates

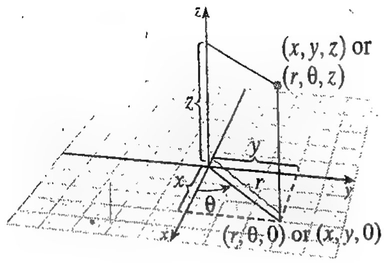


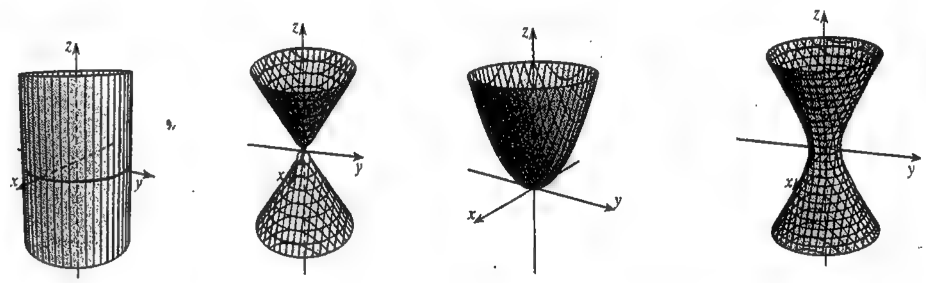
Figure 12.37 The cylindrical coordinate system

CYLINDRICAL COORDINATES

Cylindrical coordinates are a generalization of polar coordinates to surfaces in $\mathbb{R}^3$. Recall that the point P with rectangular coordinates (x, y, z) is located z units above the point $Q(x, y, 0)$ in the xy-plane (below if $z < 0$). In cylindrical coordinates, we measure the point in the xy-plane in polar coordinates, with the same z-coordinate as in the Cartesian coordinate system. These relationships are shown in Figure 12.37.

Cylindrical coordinates are convenient for representing cylindrical surfaces and surfaces of revolution for which the z-axis is the axis of symmetry. Some examples are shown in Figure 12.38.

We have the following conversion formulas, which follow directly from the rectangular-polar conversions.



	a. Cylinder	b. Cone	c. Paraboloid	d. Hyperboloid
Rectangular equation:	$x^2 + y^2 = a^2$	$x^2 + y^2 = z^2$	$x^2 + y^2 = az$	$x^2 + y^2 - z^2 = 1$
Cylindrical equation:	$r = a$	$r = z$	$r^2 = az$	$r^2 = z^2 + 1$

Figure 12.38 Surfaces with convenient cylindrical coordinates

Conversion Formulas for Rectangular/Cylindrical Coordinates

Cylindrical to rectangular: (r, θ, z) to (x, y, z)

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Rectangular to cylindrical: (x, y, z) to (r, θ, z)

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

EXAMPLE 1 Rectangular-form equation converted to cylindrical-form equation

Find an equation in cylindrical coordinates for the elliptic paraboloid $z = x^2 + 3y^2$.

Solution

We use the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} z = x^2 + 3y^2 &= (r\cos\theta)^2 + 3(r\sin\theta)^2 \\ &= r^2(\cos^2\theta + 3\sin^2\theta) \\ &= r^2[(1 - \sin^2\theta) + 3\sin^2\theta] \\ &= r^2(1 + 2\sin^2\theta) \end{aligned}$$

■

INTEGRATION WITH CYLINDRICAL COORDINATES

A triple integral $\iiint\limits_D f(x, y, z)\,dV$ can often be evaluated by transforming to cylindrical coordinates if the region of integration D is z-simple and the projection of D onto the xy-plane is a region A that can be described more naturally in terms of polar coordinates than rectangular coordinates. Suppose $f(x, y, z)$ is continuous over the region of integration D, where

$$D = \{(x, y, z) \quad \text{such that} \quad u(x, y) \le z \le v(x, y) \quad \text{for all } (x, y) \text{ in } A\}$$

Then

$$\begin{aligned} \iiint\limits_R f(x, y, z)\,dV &= \iint\limits_R \left[\int_{u(x,y)}^{v(x,y)} f(x, y, z)\,dz\right] dA \\ &= \iint\limits_D \int_{u(r\cos\theta, r\sin\theta)}^{v(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta \end{aligned}$$

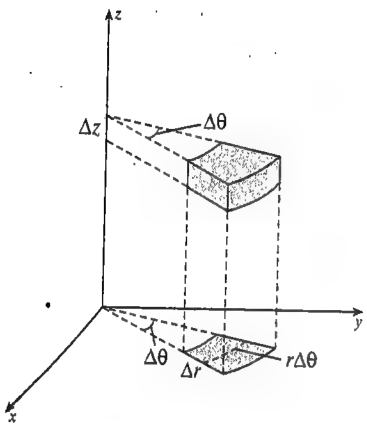


Figure 12.39 Volume element in cylindrical coordinates

since in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ (recall Theorem 12.4 and see Figure 12.39). Thus, in cylindrical coordinates $dV = r\,dr\,d\theta\,dz$. Finally, the projected region is A, where

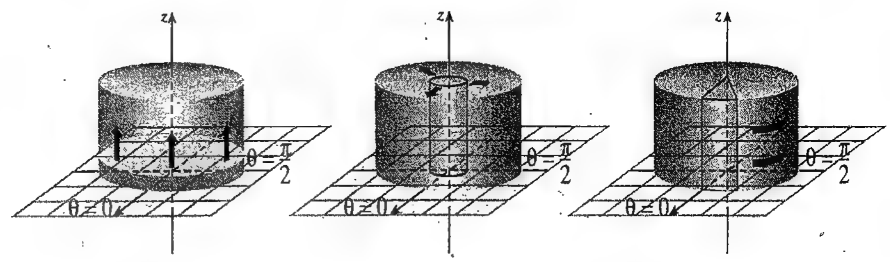
$$A = \{(r, \theta) \quad \text{such that} \quad g_1(\theta) \le r \le g_2(\theta) \quad \text{for} \quad \alpha \le \theta \le \beta\}$$

Transforming the given integral to cylindrical coordinates yields the form shown in the following box.

Triple Integral in Cylindrical Coordinates

> Let R be a solid with upper surface $z = v(r, \theta)$ and lower surface $z = u(r, \theta)$, and let D be the projection of the solid onto the xy-plane expressed in polar coordinates. Then, if $f(x, y, z)$ is continuous on R, the **triple integral of f over S** is given by
>
> $$\iiint\limits_R f(x, y, z)\,dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u(r,\theta)}^{v(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

Figure 12.40 illustrates how the region of integration D can be determined by using cylindrical coordinates.



a. Integrate with respect to z. b. Integrate with respect to r. c. Integrate with respect to θ.

Figure 12.40 Determining regions of integration in cylindrical coordinates

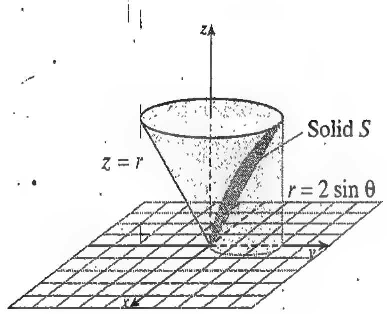


Figure 12.41 The solid bounded by $x^2+y^2=2y$, $z=\sqrt{x^2+y^2}$, and the xy-plane

EXAMPLE 2 Finding volume in cylindrical coordinates

Find the volume of the solid in the first octant that is bounded by the cylinder $x^2+y^2=2y$, the half-cone $z=\sqrt{x^2+y^2}$, and the xy-plane.

Solution

Let S be the region occupied by the solid as shown in Figure 12.41. This surface is most easily described in cylindrical coordinates.

Cylinder:	Cone:
$x^2+y^2=2y$	$z=\sqrt{x^2+y^2}$
$r^2=2r\sin\theta$	$z=r$
$r=2\sin\theta$	

Since the region S lies in the first octant, we have $0\le\theta\le\frac{\pi}{2}$, so S may be described by

$$0\le z\le r \qquad 0\le r\le 2\sin\theta \qquad 0\le\theta\le\frac{\pi}{2}$$

$$\begin{aligned}
V &= \iiint\limits_R dV = \iint\limits_D \int_{u(r,\theta)}^{v(r,\theta)} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{r} r\,dz\,dr\,d\theta \\
&= \int_0^{\pi/2}\int_0^{2\sin\theta} r^2\,dr\,d\theta = \int_0^{\pi/2} \frac{r^3}{3}\Big|_{r=0}^{r=2\sin\theta} d\theta = \frac{8}{3}\int_0^{\pi/2}\sin^3\theta\,d\theta \\
&= \frac{8}{3}\int_0^{\pi/2}(1-\cos^2\theta)\sin\theta\,d\theta \\
&= \frac{8}{3}\left[-\cos\theta+\frac{\cos^3\theta}{3}\right]\Bigg|_0^{\pi/2} = \frac{16}{9}
\end{aligned}$$

■

EXAMPLE 3 Centroid in cylindrical coordinates

A homogeneous solid S with constant density δ is bounded below by the xy-plane, on the sides by the cylinder $x^2+y^2=a^2 (a>0)$, and above by the surface $z=x^2+y^2$. Find the centroid of the solid.

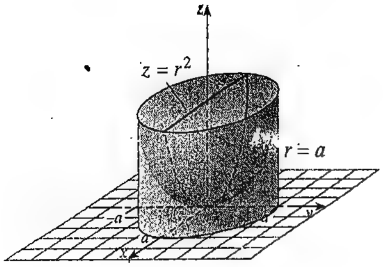


Figure 12.42 The solid bounded by $x^2 + y^2 = a^2$ and $z = x^2 + y^2$

Solution

The solid S is shown in Figure 12.42. Because the solid is bounded by a cylinder, we will carry out the integration in cylindrical coordinates.

Cylinder:	Paraboloid:
$x^2 + y^2 = a^2$	$z = x^2 + y^2$
$r^2 = a^2$	$z = r^2$
$r = a \quad (a > 0)$	

Let $(\overline{x}, \overline{y}, \overline{z})$ denote the centroid. Using symmetry and the fact that the density function is constant, we have $\overline{x} = \overline{y} = 0$. Let m denote the mass of S. Since the projected region is $r = a$ for $0 \leq \theta \leq 2\pi$, we find that

$$\overline{z} = \frac{M_{xy}}{m} = \frac{\iiint\limits_S zr\,dz\,dr\,d\theta}{\iiint\limits_S r\,dz\,dr\,d\theta} = \frac{\int_0^{2\pi}\int_0^a\int_0^{r^2} zr\,dz\,dr\,d\theta}{\int_0^{2\pi}\int_0^a\int_0^{r^2} r\,dz\,dr\,d\theta} = \frac{\frac{\pi}{6}a^6}{\frac{\pi}{2}a^4} = \frac{a^2}{3}$$

Verify the details of the integration. The centroid is $\left(0, 0, \frac{a^2}{3}\right)$. ■

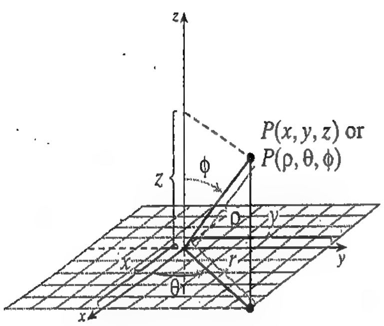


Figure 12.43 The spherical coordinate system

SPHERICAL COORDINATES

In **spherical coordinates** we label a point P by a triple (ρ, θ, ϕ), where ρ, θ, and ϕ are numbers determined as follows (refer to Figure 12.43):

ρ = the distance from the origin to the point P; we require $\rho \geq 0$.

θ = the polar angle (as in polar coordinates); we require $0 \leq \theta \leq 2\pi$.

ϕ = the angle measured down from the positive z-axis to the ray from the origin through P; we require $0 \leq \phi \leq \pi$.

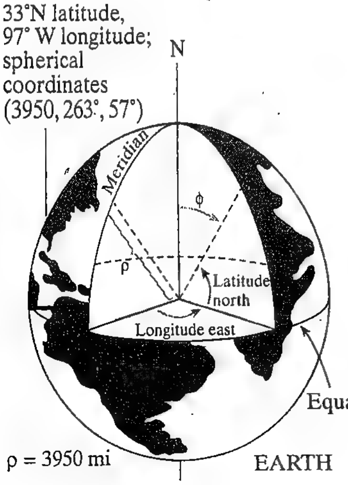
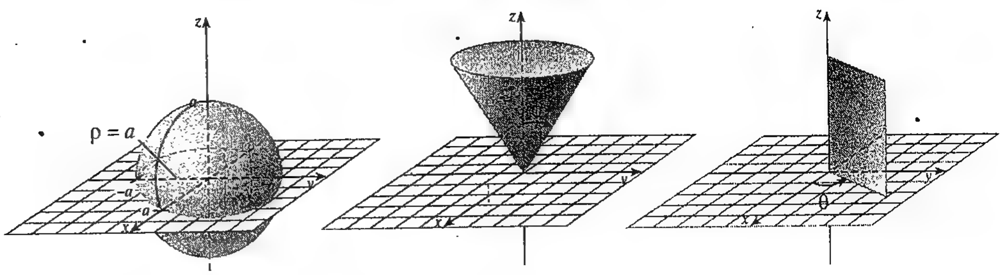


Figure 12.44 Spherical coordinates on the earth's surface

You might recognize that spherical coordinates are related to the longitude and latitude coordinates used in navigation. To be more specific, consider a rectangular coordinate system with the origin at the center of the earth, with the positive z-axis passing through the north pole and the xz-plane passing through the prime meridian. Then, a particular location on the surface is denoted by (ρ, θ, ϕ), where ρ is the distance from the center of the earth, θ is the longitude, and $\frac{\pi}{2} - \phi$ is the latitude (since latitude is the angle up from the equator). For example, Dallas-Fort Worth has coordinates $(\rho, \theta, \phi) = (3950, 263°, 57°)$, as shown in Figure 12.44.

Spherical coordinates are desirable when representing spheres, cones, or certain planes. Some examples are shown in Figure 12.45.

We use the relationships in Figure 12.45 to obtain the remaining conversion formulas. All of these formulas can be derived by considering the relationships among the variables as shown in Figure 12.46.



a. Sphere, $\rho = a\ (a > 0)$ **b.** Half-cone, $\phi = a\ (0 < a < \frac{\pi}{2})$ **c.** Vertical half-plane, $\theta = a$

Figure 12.45 Surfaces with convenient spherical coordinates

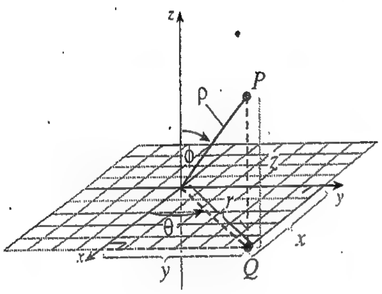


Figure 12.46 Relations among coordinate systems

Conversion Formulas Involving Spherical Coordinates

Spherical to rectangular: (ρ, θ, ϕ) to (x, y, z)

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Rectangular to spherical: (x, y, z) to (ρ, θ, ϕ)

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\tan\theta = \frac{y}{x}$$
$$\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Spherical to cylindrical: (ρ, θ, ϕ) to (r, θ, z)

$$r = \rho \sin\phi$$
$$\theta = \theta$$
$$z = \rho \cos\phi$$

Cylindrical to spherical: (r, θ, z) to (ρ, θ, ϕ)

$$\rho = \sqrt{r^2 + z^2}$$
$$\theta = \theta$$
$$\phi = \cos^{-1}\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$$

EXAMPLE 4 Converting rectangular-form equations to spherical-form

Rewrite each of the given equations in spherical form.

a. the sphere $x^2 + y^2 + z^2 = a^2 \quad (a > 0)$ **b.** the paraboloid $z = x^2 + y^2$

Solution

a. Because $\rho = \sqrt{x^2 + y^2 + z^2}$, we see $x^2 + y^2 + z^2 = \rho^2$, so we can write

$$\rho^2 = a^2$$
$$\rho = a \qquad \text{Because } \rho \geq 0.$$

b.

$$\begin{aligned} z &= x^2 + y^2 \\ \rho\cos\phi &= (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 \\ \rho\cos\phi &= \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta \\ &= \rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) \\ \rho &= \frac{\cos\phi}{\sin^2\theta} = \cot\phi\csc\phi \end{aligned}$$

■

INTEGRATION WITH SPHERICAL COORDINATES

For a solid S in spherical coordinates, the fundamental element of volume is a spherical "wedge" bounded in such a way that

$$\rho_1 \le \rho \le \rho_1 + \Delta\rho \qquad \phi_1 \le \phi \le \phi_1 + \Delta\phi, \qquad \theta_1 \le \theta \le \theta_1 + \Delta\theta$$

This "wedge" is shown in Figure 12.47 and has volume $\rho^2 \sin\phi\ \Delta\rho\ \Delta\phi\ \Delta\theta$.

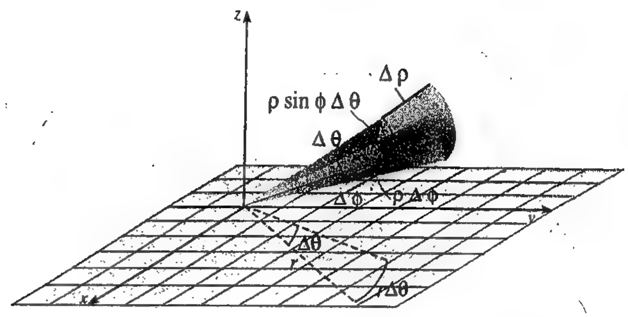


Figure 12.47 A spherical wedge

In Section 12.6 we show that the element of volume in spherical coordinates is

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Using this formula, we can form partitions and take a limit of a Riemann sum as the partitions are refined to obtain the integral form shown in the following box.

Triple Integral in Spherical Coordinates

If f is continuous on the bounded solid region S, then the **triple integral of f over S** is given by

$$\iiint_S f(x, y, z)\, dV$$
$$= \iiint_{S'} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

where S' is the region S expressed in spherical coordinates.

EXAMPLE 5 Volume of a sphere

In geometry, it is shown that a sphere of radius R has volume $V = \frac{4}{3}\pi R^3$. Verify this formula using integration.

Solution

It seems clear that we should work in spherical coordinates with the origin of the coordinate system at the center of the sphere, because the equation of the sphere is $\rho = R$ for $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

$$V = \iiint_S dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3}\sin\phi\Big|_{\rho=0}^{\rho=R} d\phi \, d\theta = \frac{R^3}{3}\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= \frac{R^3}{3}\int_0^{2\pi} (-\cos\phi)|_0^{\pi} \, d\theta = \frac{R^3}{3}\int_0^{2\pi} 2 \, d\theta$$

$$= \frac{2R^3}{3}[\theta]|_0^{2\pi} = \frac{2R^3}{3}(2\pi) = \frac{4}{3}\pi R^3$$

■

EXAMPLE 6 Deciding which coordinate system to apply to an integral

Evaluate the integral

$$I = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} z \, dz \, dy \, dx$$

using the given rectangular form, or by transforming to either cylindrical or spherical coordinates.

Solution

The region of integration is D, where

$$D = \Big\{(x, y, z) \quad \text{such that } x^2 + y^2 \le z \le \sqrt{2 - x^2 - y^2} \text{ for}$$
$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \text{ and } -1 \le x \le 1\Big\}$$

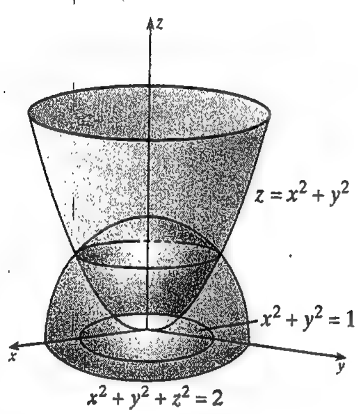


Figure 12.48 The solid D and the projected region in the xy-plane

Geometrically, D is the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$. The bounding surfaces intersect where

$$z = x^2 + y^2 \quad \text{and} \quad x^2 + y^2 + z^2 = 2$$

so

$$\begin{aligned} z + z^2 &= 2 && \textbf{Substitute the first into the second.} \\ (z-1)(z+2) &= 0 \\ z &= 1 && \textbf{Reject } z = -2, \textbf{ since } z \ge 0. \end{aligned}$$

It follows that the projected region in the xy-plane is the disk $x^2 + y^2 \le 1$ (see Figure 12.48).

It may seem that transforming to spherical coordinates may result in a less complicated integration, since the upper boundary of D has the simple form $\rho = \sqrt{2}$, but the lower boundary $z = x^2 + y^2$ has the form

$$\begin{aligned} z &= x^2 + y^2 = r^2 \\ \rho\cos\phi &= (\rho\sin\phi)^2 \\ \rho &= \frac{\cos\phi}{\sin^2\phi} = \cot\phi\csc\phi \end{aligned}$$

Moreover, for $0 \le \phi \le \frac{\pi}{4}$, we have $0 \le \rho \le \sqrt{2}$, but $0 \le \rho \le \cot\phi \csc\phi$ for $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$, which means we need two integrals to evaluate the integral in spherical coordinates:

$$I = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$+ \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\cot\phi\csc\phi} (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

However, if we use cylindrical coordinates, the integral has the relatively simple form

$$I = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} zr \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r\left[\tfrac{1}{2}z^2\right]\Big|_{r^2}^{\sqrt{2-r^2}} dr \, d\theta$$
$$= \int_0^{2\pi} \int_0^1 \tfrac{1}{2}r\left[(2-r^2) - r^4\right] dr \, d\theta = \tfrac{1}{2}\int_0^{2\pi} \left[r - \tfrac{1}{4}r^3 - \tfrac{1}{6}r^6\right]\Big|_0^1 d\theta$$
$$= \tfrac{1}{2}\left(\tfrac{7}{12}\right)\int_0^{2\pi} d\theta = \tfrac{7}{24}(2\pi) = \tfrac{7\pi}{12}$$

Guidelines Qs 1-22, 24

12.5 PROBLEM SET (HW③ B38-pg 96)

A 1. **WHAT DOES THIS SAY?** Compare and constrast the rectangular, cylindrical, and spherical coordinate systems.

2. **Exploration Problem** Suppose you need to evaluate a particular triple integral. Discuss some criteria for choosing a coordinate system.

In Problems 3–4, convert from rectangular coordinates to
a. cylindrical *b. spherical*

3. $(0, 4, \sqrt{3})$
4. $(1, 2, 3)$

In Problems 5–6, convert from cylindrical coordinates to
a. rectangular *b. spherical*

5. $(4, \frac{\pi}{6}, -2)$
6. $(2, \frac{\pi}{4}, \pi)$

In Problems 7–8, convert from spherical coordinates to
a. rectangular *b. cylindrical*

7. $(2, \frac{\pi}{6}, \frac{2\pi}{3})$
8. (π, π, π)

Evaluate each iterated integral in Problems 9–13.

9. $\displaystyle\int_0^{\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} r\sin\theta \, dz \, dr \, d\theta$

10. $\displaystyle\int_0^{\pi/2} \int_0^{2\pi} \int_0^2 \cos\phi\sin\phi \, d\rho \, d\theta \, d\phi$

11. $\displaystyle\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

12. $\displaystyle\int_{-\pi/4}^{\pi/3} \int_0^{\sin\theta} \int_0^{4\cos\theta} r \, dz \, dr \, d\theta$

13. $\displaystyle\int_0^{\pi/3} \int_0^{\cos\theta} \int_0^{\phi} \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta$

B 14. The point (x, y, z) lies on an ellipsoid if

Check P684

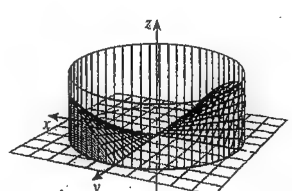
$$x = aR\sin\phi\cos\theta$$
$$y = bR\sin\phi\sin\theta$$
$$z = cR\cos\phi$$

for a constant R. Find an equation for this ellipsoid in rectangular coordinates.

15. Use cylindrical coordinates to compute the integral

$$\iiint_R xy \, dx \, dy \, dz$$

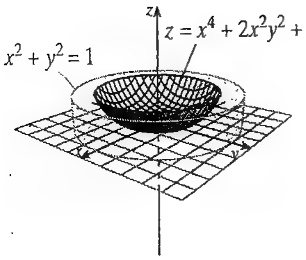
where R is the cylindrical solid $x^2 + y^2 \le 1$ with $0 \le z \le 1$.



16. Use cylindrical coordinates to compute the integral

$$\iiint_R (x^4 + 2x^2y^2 + y^4) \, dx \, dy \, dz$$

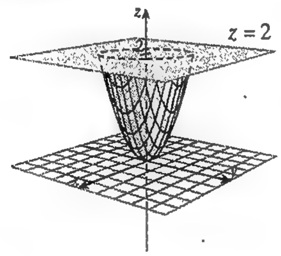
where R is the cylindrical solid $x^2 + y^2 \le a^2$ with $0 \le z \le \frac{1}{\pi}$.



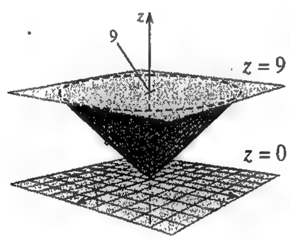
17. Use cylindrical coordinates to compute the integral

$$\iiint_D z(x^2 + y^2)^{-1/2}\, dx\, dy\, dz$$

where D is the solid bounded above by the plane $z = 2$ and below by the surface $2z = x^2 + y^2$.



18. Find the centroid of the solid bounded by the surface $z = \sqrt{x^2 + y^2}$ and the plane $z = 9$.



The average value of a function $f(x, y, z)$ over a solid region D is given by

$$\text{AVERAGE VALUE} = \frac{1}{\text{VOLUME OF } D} \iiint_D f(x, y, z)\, dV$$

Use this definition in Problem 19.

19. Find the average value of the function $f(x, y, z) = x + y + z$ over the sphere $x^2 + y^2 + z^2 = 4$.

20. Evaluate $\displaystyle\iiint_R \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz$ where R is defined by $x^2 + y^2 + z^2 \le 2$.

21. Evaluate $\displaystyle\iiint_D z^2\, dx\, dy\, dz$ where D is the solid hemisphere $x^2 + y^2 + z^2 \le 1, z \ge 0$.

22. Evaluate $\displaystyle\iiint_D \frac{dx\, dy\, dz}{\sqrt{x^2 + y^2 + z^2}}$ where D is the solid sphere $x^2 + y^2 + z^2 \le 3$.

Find the volume of the solid D given in Problems 23–25 by using integration in any convenient system of coordinates.

23. D is bounded by the paraboloid $z = 1 - 4(x^2 + y^2)$ and the xy-plane. One u gor is cack (pg 93)

24. D is the intersection of the solid sphere $x^2 + y^2 + z^2 \le 9$ and the solid cylinder $x^2 + y^2 \le 1$.

25. D is the region that remains in the spherical solid $\rho \le 4$ after the solid cone $\phi \le \frac{\pi}{6}$ has been removed.

26. How much volume remains from a spherical ball of radius a when a cylindrical hole of radius b $(0 < b < a)$ is bored out of its center? (See Figure 12.49.)

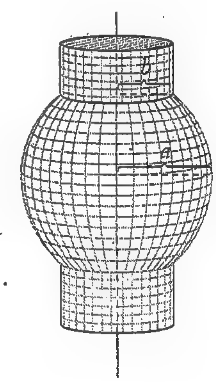


Figure 12.49 Boring a hole in a spherical ball

12.6 Jacobians: Change of Variables

IN THIS SECTION change of variables in a double integral, change of variables in a triple integral

CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

When the change of variable $x = g(u)$ is made in the single integral, we know

$$\int_a^b f(x)\, dx = \int_c^d f(g(u))g'(u)\, du$$

where the limits of integration c and d satisfy $a = g(c)$ and $b = g(d)$. By changing variables in a double integral $\iint_D f(x, y)\, dA$, we want to transform the integrand

$f(x, y)$ and the region of integration D so that the modified integral is easier to evaluate than the original. In general, this process involves introducing a "mapping factor" analogous to the term $g'(u)$ in the single-variable case. This factor is called a *Jacobian* in honor of the German mathematician Karl Gustav Jacobi (1804–1851; see Historical Quest Problem 31), who made the first systematic study of change of variables in multiple integrals in the middle of the nineteenth century.

Suppose we want to evaluate the double integral $\iint\limits_D f(x, y)\,dy\,dx$ by converting it to an equivalent integral involving the variables u and v. This conversion is determined by a transformation (function) T that maps the uv-plane onto the xy-plane. If $T(u, v) = (x, y)$, then (x, y) is the *image* of (u, v) under T, and if no two points in the uv-plane map into the same point (x, y) in the xy-plane, then T is *one-to-one*. In this case, it may be possible to solve the equations $x = x(u, v)$ and $y = y(u, v)$ for u and v in terms of x and y to obtain the equations $u = u(x, y)$ and $v = v(x, y)$, which defines a **transformation** from the xy-plane back to the uv-plane, called the **inverse transformation** of T, and is denoted by T^{-1}. This terminology is illustrated in Figure 12.50. The basic result we will use for changing variables in a double integral is stated in the following theorem.

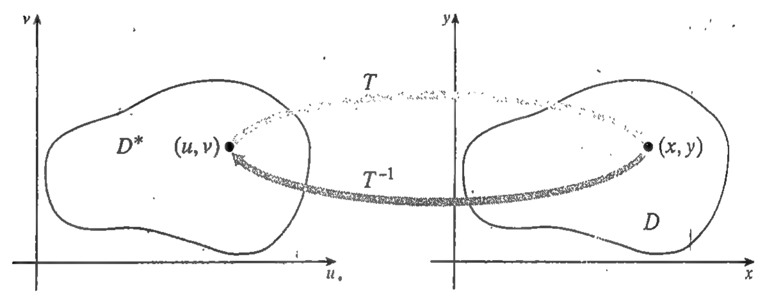


Figure 12.50 A one-to-one transformation T and its inverse T^{-1}

***THEOREM 12.7* Change of variables in a double integral**

Let f be a continuous function on a region D in the xy-plane, and let T be a one-to-one transformation that maps the region D^* in the uv-plane onto D under the change of variables $x = g(u, v)$, $y = h(u, v)$, for functions g and h continuously differentiable function in D^*. Then

$$\iint\limits_D f(x, y)\,dy\,dx = \iint\limits_{D^*} f[g(u, v), h(u, v)]\,|J(u, v)|\,du\,dv$$

where

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

is nonzero and does not change sign on D^*. The mapping factor $J(u, v)$ is called the **Jacobian** and is also denoted by $\dfrac{\partial(x, y)}{\partial(u, v)}$.

Proof A formal proof is a matter for advanced calculus, but a geometric argument for this theorem is presented in Appendix B. ❑

EXAMPLE 1 Finding a Jacobian

Find the Jacobian for the change of variables from rectangular to polar coordinates, namely, $x = r\cos\theta$ and $y = r\sin\theta$.

Solution

The Jacobian of the change of variables is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r \qquad \blacksquare$$

This result justifies the formula we have previously used:

$$\iint_D f(x,y)\,dy\,dx = \iint_{D^*} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

where D^* is the region in the (r, θ) plane that is mapped into the region D in the (x, y) plane by the polar transformation $x = r\cos\theta$, $y = r\sin\theta$, as illustrated in Figure 12.51.

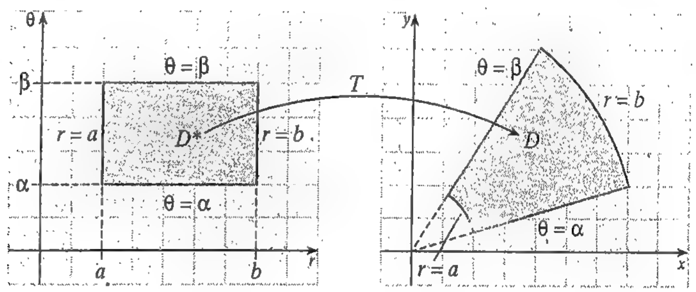


Figure 12.51 Transformation of a region D^* by $T: x = r\cos\theta$, $y = r\sin\theta$

Sometimes it is easier to express u and v in terms of x and y, and to compute the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$. The Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ needed for converting the integral $\iint\limits_D f(x,y)\,dy\,dx$ can then be computed by the formula

$$\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)} = 1$$

(see Problem 32). This procedure is illustrated by the following example.

EXAMPLE 2 Finding the Jacobian when $u = u(x, y)$ and $v = v(x, y)$ are given

If $u = xy$ and $v = x^2 - y^2$, express the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ in terms of u and v.

Solution

It is not easy to express x and y in terms of u and v. However,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2mm] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2y^2 - 2x^2 = -2(x^2 + y^2)$$

Since

$$(x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2$$

$$v^2 + 4u^2 = (x^2 + y^2)^2 \quad \textbf{Substitute } v = x^2 - y^2 \textbf{ and } u^2 = x^2y^2.$$

$$x^2 + y^2 = \sqrt{4u^2 + v^2} \quad \textbf{Note: } x^2 + y^2 \textbf{ is nonnegative.}$$

Therefore, since $\dfrac{\partial(x, y)}{\partial(u, v)} \dfrac{\partial(u, v)}{\partial(x, y)} = 1$, we have

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\dfrac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{-2(x^2 + y^2)} = \frac{-1}{2\sqrt{4u^2 + v^2}} \quad \blacksquare$$

EXAMPLE 3 Calculating a double integral by changing variables

Compute $\displaystyle\iint\limits_D \left(\frac{x - y}{x + y}\right)^4 dy\, dx$, where D is the triangular region bounded by the line $x + y = 1$ and the coordinate axes.

Solution

This is a rather difficult computation if no substitution is made. The form of the integral suggests we make the substitution

$$u = x - y, \qquad v = x + y$$

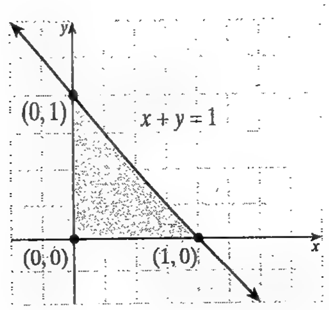
Solving for x and y, we obtain

$$x = \tfrac{1}{2}(u + v), \qquad y = \tfrac{1}{2}(v - u)$$

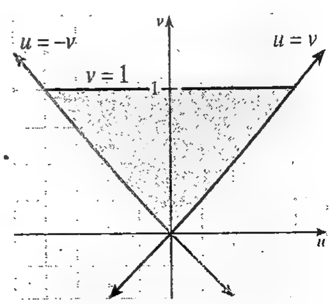
and the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial}{\partial u}\left(\dfrac{u + v}{2}\right) & \dfrac{\partial}{\partial v}\left(\dfrac{u + v}{2}\right) \\ \dfrac{\partial}{\partial u}\left(\dfrac{v - u}{2}\right) & \dfrac{\partial}{\partial v}\left(\dfrac{v - u}{2}\right) \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

To find the image D^* of D in the uv-plane, note that the boundary lines $x = 0$ and $y = 0$ for D map into the lines $u = -v$ and $u = v$, respectively, while $x + y = 1$ maps into $v = 1$. Therefore, the transformed region of integration D^* is the triangular region shown in Figure 12.52b, with vertices $(0, 0)$, $(1, 1)$, and $(-1, 1)$.



a. The region of integration D



b. The transformed region D^*

Figure 12.52 Transformation of D to D^*

We now evaluate the given integral:

$$\iint_D \left(\frac{x-y}{x+y}\right)^4 dy\,dx = \iint_{D^*} \left(\frac{u}{v}\right)^4 \left|\frac{1}{2}\right| du\,dv = \frac{1}{2}\int_0^1 \int_{-v}^{v} u^4 v^{-4}\,du\,dv = \frac{1}{10}$$ ■

In Example 3, the change of variables was chosen to simplify the integrand, but sometimes it is useful to introduce a change of variables that simplifies the region of integration.

EXAMPLE 4 Change of variables to simplify a region

Find the area of the region E bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

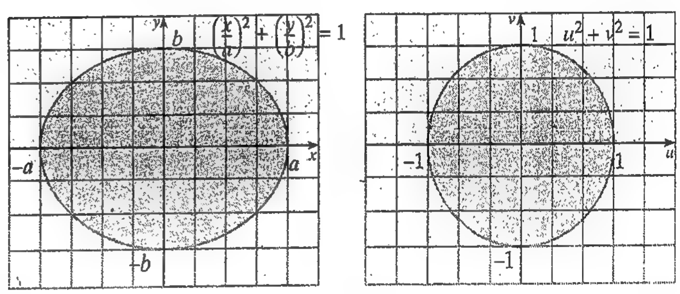
Solution

The area is given by the integral

$$A = \iint_E dy\,dx$$

where E is the region shown in Figure 12.53a. Because E can be represented by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \le 1$$



a. The elliptical region E

b. The transformed region C is a circular disk.

Figure 12.53 Transformation of the ellipse E to the circle C

we consider the substitution $u = \frac{x}{a}$ and $v = \frac{y}{b}$, which will map the elliptical region E onto the circular disk

$$C\colon u^2 + v^2 \le 1$$

as shown in Figure 12.53b. Then $x = au$ and $y = bv$, and the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial}{\partial u}(au) & \frac{\partial}{\partial v}(au) \\ \frac{\partial}{\partial u}(bv) & \frac{\partial}{\partial v}(bv) \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

Because $ab > 0$, we have $|ab| = ab$, and the area of E is given by

$$\begin{aligned}\iint\limits_E dy\,dx &= \iint\limits_C ab\,du\,dv \\ &= ab\iint\limits_C du\,dv \\ &= ab\iint\limits_C 1\,dA \\ &= ab\left[\pi(1)^2\right] \qquad \text{Because } C \text{ is a circle of radius 1} \\ &= \pi ab\end{aligned}$$

■

EXAMPLE 5 Using a change of variables to find a centroid

Find the centroid of the region D^* in the xy-plane that is bounded by the lines $y = \frac{1}{4}x$ and $y = \frac{5}{2}x$ and the hyperbolas $xy = 1$ and $xy = 5$.

Solution

If A is the area of D^*, the centroid is $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{1}{A}\iint\limits_{D^*} x\,dA \quad \text{and} \quad \overline{y} = \frac{1}{A}\iint\limits_{D^*} y\,dA$$

We wish to find a transformation that simplifies the boundary curves of the region of integration, so we let $u = \dfrac{y}{x}$ and $v = xy$ and the boundaries of the transformed region D become

$$\begin{array}{cccc} u = \frac{1}{4} & u = \frac{5}{2} & v = 1 & v = 5 \\ \left(y = \frac{1}{4}x\right) & \left(y = \frac{5}{2}x\right) & (xy = 1) & (xy = 5) \end{array}$$

The regions D^* and D are shown in Figure 12.54.

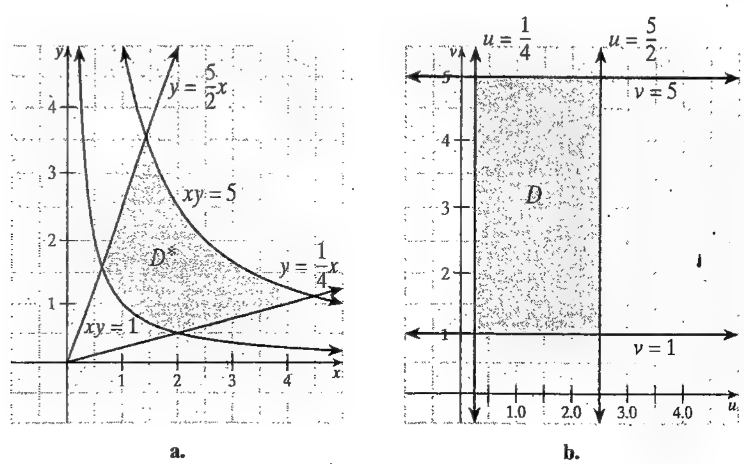


Figure 12.54 The regions D^* and D

Solving the equations $u = y/x$ and $v = xy$ for x and y, we obtain $x = \sqrt{v/u}$ and $y = \sqrt{uv}$, so the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{1}{2}\dfrac{1}{u}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2}\dfrac{1}{\sqrt{uv}} \\ \dfrac{1}{2}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2}\sqrt{\dfrac{u}{v}} \end{vmatrix} = \frac{-1}{4u} - \frac{1}{4u} = \frac{-1}{2u}$$

The area of the region D^* is

$$\begin{aligned} A &= \iint_{D^*} dy\,dx \\ &= \int_1^5 \int_{1/4}^{5/2} \left| \frac{-1}{2u} \right| du\,dv \\ &= \frac{1}{2} \int_1^5 \ln u \Big|_{1/4}^{5/2} dv \\ &= \frac{1}{2}\left[\ln\frac{5}{2} - \ln\frac{1}{4}\right] \int_1^5 dv \\ &= \frac{1}{2} \ln\left(\frac{5/2}{1/4}\right) v\Big|_1^5 = 2\ln 10 \end{aligned}$$

To find the centroid $(\overline{x}, \overline{y})$, we compute

$$\overline{x} = \frac{1}{A}\iint_{D^*} x\,dA = \frac{1}{2\ln 10}\int_1^5 \int_{1/4}^{5/2} \sqrt{\frac{v}{u}} \left| \frac{-1}{2u} \right| du\,dv \approx 2.0154$$

$$\overline{y} = \frac{1}{A}\iint_{D^*} y\,dA = \frac{1}{2\ln 10}\int_1^5 \int_{1/4}^{5/2} \sqrt{uv} \left| \frac{-1}{2u} \right| du\,dv \approx 1.5933$$

Thus, the centroid is at approximately (2.0154, 1.5933) in the xy-plane. ■

CHANGE OF VARIABLES IN A TRIPLE INTEGRAL

The change of variables formula for triple integrals is similar to the one given for double integrals. Let T be a change of variables that maps a region R^* in uvw-space onto a region R in xyz-space, where

$$T: x = x(u, v, w) \quad y = y(u, v, w) \quad z = z(u, v, w)$$

Then the Jacobian of T is the determinant

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

and the change of variable yields

$$\iiint_R f(x, y, z)\,dx\,dy\,dz$$

$$= \iiint_{R^*} f[x(u, v, w), y(u, v, w), z(u, v, w)] \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\,dv\,dw$$

EXAMPLE 6 Formula for integrating with spherical coordinates

Obtain the formula for converting a triple integral in rectangular coordinates to one in spherical coordinates.

Solution

The conversion formulas from rectangular coordinates to spherical coordinates are

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

The Jacobian of this transformation is

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \frac{\partial}{\partial \rho}(\rho \sin\phi \cos\theta) & \frac{\partial}{\partial \theta}(\rho \sin\phi \cos\theta) & \frac{\partial}{\partial \phi}(\rho \sin\phi \cos\theta) \\ \frac{\partial}{\partial \rho}(\rho \sin\phi \sin\theta) & \frac{\partial}{\partial \theta}(\rho \sin\phi \sin\theta) & \frac{\partial}{\partial \phi}(\rho \sin\phi \sin\theta) \\ \frac{\partial}{\partial \rho}(\rho \cos\phi) & \frac{\partial}{\partial \theta}(\rho \cos\phi) & \frac{\partial}{\partial \phi}(\rho \cos\phi) \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}$$

$$= -\rho^2 \sin\phi$$

SMH See Problem 91 of Problem Set 2 of the Student Mathematics Handbook.

Since $0 \le \phi \le \pi$, we have $\sin\phi \ge 0$, so

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = |-\rho^2 \sin\phi| = \rho^2 \sin\phi$$

$$\iiint_R f(x, y, z)\,dz\,dx\,dy$$

$$= \iiint_{R^*} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi \qquad \blacksquare$$

uw(1) → P 109 – 128 Guidelines

12.6 PROBLEM SET

Qs 1–13, 18–28, 30

A *Find the Jacobian* $\dfrac{\partial(x, y)}{\partial(u, v)}$ *or* $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ *in Problems 1–8.*

1. $x = u + 2v$, $y = 3u - 4v$
2. $x = u^2v^2$, $y = v^2 - u^2$
3. $x = e^{u+v}$, $y = e^{u-v}$
4. $x = e^u \sin v$, $y = e^u \cos v$
5. $x = \dfrac{v}{u^2 + v^2}$, $y = \dfrac{u}{u^2 + v^2}$
6. $x = u + v - w$; $y = 2u - v + 3w$; $z = -u + 2v - w$
7. $x = u \cos v$, $y = u \sin v$, $z = we^{uv}$

8. $x = \frac{u}{v},\ y = \frac{v}{w},\ z = \frac{w}{u}$

Find the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ in Problems 9–11 either by solving the given equations for x and y or by first computing $\frac{\partial(u, v)}{\partial(x, y)}$. Express your answers in terms of u and v.

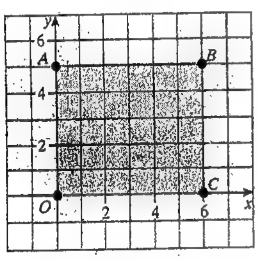
9. $u = ye^{-x},\ v = e^x$

10. $u = xy,\ v = \frac{y}{x}$ for $x > 0, y > 0$

11. $u = \frac{x}{x^2 + y^2},\ v = \frac{y}{x^2 + y^2}$

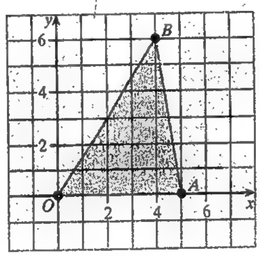
A region R is given in Problems 12–15. Sketch the corresponding region R^* in the uv-plane using the given transformations.

12.



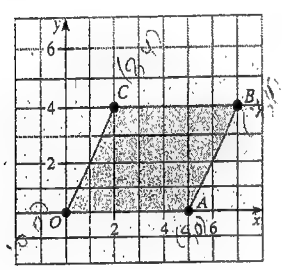
$u = x + y,\ v = x - y$

13.



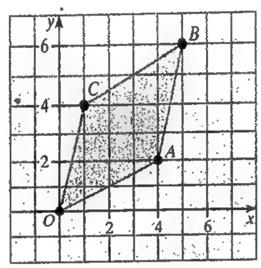
$u = 2x,\ v = x + y$

14.



$u = x^2 - y^2,\ v = 2xy$

15.



$u = x^2,\ v = x + y$

16. Suppose the uv-plane is mapped onto the xy-plane by the equations $x = u(1 - v),\ y = uv$. Express $dx\,dy$ in terms of $du\,dv$.

17. Suppose the uv-plane is mapped onto the xy-plane by the equations $x = u^2 - v^2,\ y = 2uv$. Express $dx\,dy$ in terms of $du\,dv$.

Use a suitable change of variables to find the area of the region specified in Problems 18–19.

18. The region R bounded by the hyperbolas $xy = 1$ and $xy = 4$, and the lines $y = x$ and $y = 4x$

19. The region R bounded by the parabolas $y = x^2$, $y = 4x^2$, $y = \sqrt{x}$, and $y = \frac{1}{2}\sqrt{x}$

Let D be the region in the xy-plane that is bounded by the coordinate axes and the line $x + y = 1$. Use the change of variables $u = x - y$, $v = x + y$ to compute the integrals given in Problems 20–22.

20. $\iint\limits_D \left(\frac{x - y}{x + y}\right)^5 dy\,dx$

21. $\iint\limits_D (x - y)^5(x + y)^3\,dy\,dx$

22. $\iint\limits_D (x - y)e^{x^2+y^2}\,dy\,dx$

B 23. Evaluate

$$\iint\limits_R e^{(2y-x)/(y+2x)}\,dA$$

where R is the trapezoid with vertices $(0, 2)$, $(1, 0)$, $(4, 0)$, and $(0, 8)$.

24. Let R be the region in the xy-plane that is bounded by the parallelogram with vertices $(0, 0)$, $(1, 1)$, $(2, 1)$ and $(1, 0)$. Use the linear transformation $x = u + v$, $y = v$ to compute $\iint\limits_R (2x - y)\,dy\,dx$.

25. Use a suitable linear transformation $u = ax + by$, $v = rx + sy$ to evaluate the integral

$$\iint\limits_R \left(\frac{x + y}{2}\right)^2 e^{(y-x)/2}\,dy\,dx$$

where R is the region inside the square with vertices $(0, 0)$, $(1, 1)$, $(0, 2)$, and $(-1, 1)$.

26. Use the change of variable $x = ar\cos\theta$, $y = br\sin\theta$ to evaluate

$$\iint\limits_{D^*} \exp\left(-\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dy\,dx$$

where D^* is the region bounded by the quarter ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{in the first quadrant}$$

27. Evaluate

$$\iint\limits_{D^*} e^{-(4x^2+5y^2)}\,dA$$

where D^* is the elliptical disk $\frac{x^2}{5} + \frac{y^2}{4} \le 1$.

28. Use the change of variables $x = au$, $y = bv$, $z = cw$ to find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

29. Evaluate

$$\iint\limits_R \ln\left(\frac{x - y}{x + y}\right) dy\,dx$$

where R is the triangular region with vertices $(1, 0)$, $(4, -3)$, and $(4, 1)$.

30. Find the Jacobian of the cylindrical coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

C 31. **HISTORICAL QUEST** Karl G. Jacobi was a gifted teacher and one of Germany's most distinguished mathematicians during the first half of the nineteenth century. He made major contributions to the theory of elliptic functions, but his work with functional determinants is what secured his place in history. In 1841 he published a long memoir called "De determinantibus functionalibus," devoted to what we today call the Jacobian and pointing out that this determinant is in many ways the multivariable analogue to the differential of a single variable. The memoir was published in what is usually known as

KARL G. JACOBI
1804–1851

Crelle's Journal, one of the first journals devoted to serious mathematics. It was begun in 1826 by August Crelle (1780–1855) with the title *Journal für die reine und angewandte Mathematik*. ■

For this Quest, set up an integral for the circumference of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This is an example of an elliptic integral.

32. Let $T: x=x(u,v), y=y(u,v)$ be a one-to-one transformation on a set D so that T^{-1} has the form $u=u(x,y), v=v(x,y)$. Use the multiplicative property of determinants, along with the chain rule, to show that

$$\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)}=1$$

You may assume that all the necessary partial derivatives exist.

SMH Properties of determinants are found in Section 2.8 of the Student Mathematics Handbook.

CHAPTER 12 REVIEW

Proficiency Examination

PRACTICE PROBLEMS

1. Evaluate $\int_0^{\pi/3}\int_0^{\sin y} e^{-x}\cos y\,dx\,dy$.
2. Use a double integral to compute the area of the region R that is bounded by the x-axis and the parabola $y=9-x^2$.
3. Use polar coordinates to evaluate

$$\int_0^1\int_0^{\sqrt{1-x^2}}\cos(x^2+y^2)\,dy\,dx$$

4. Use a triple integral to find the volume of the tetrahedron that is bounded by the coordinate planes and the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

with $a>0, b>0, c>0$.

5. Use a linear change of variables to evaluate the double integral

$$\iint_R (x+y)e^{x-2y}\,dy\,dx$$

where R is the triangular region with vertices $(0,0)$, $(2,0)$, $(1,1)$.

Supplementary Problems

Evaluate the integrals given in Problems 1–8.

1. $\iint_D x^3\,dA$, where D is the shaded portion shown in Figure 12.55.

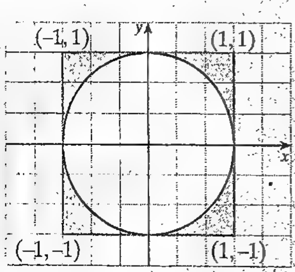


Figure 12.55 Problem 1

2. $\iint_D xy^2\,dA$, where D is the shaded portion shown in Figure 12.56.

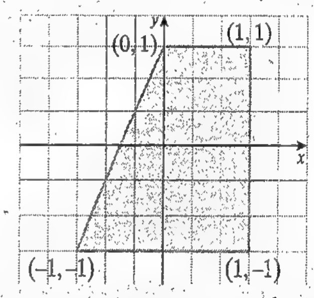


Figure 12.56 Problem 2

3. $\iint_D \exp\left(\frac{y-x}{y+x}\right)dy\,dx$, where D is the triangular region with vertices $(0,0)$, $(2,0)$, $(0,2)$. *Hint:* Make a suitable change of variables.
4. $\iiint_D \sqrt{x^2+y^2+z^2}\,dV$, where D is the portion of the solid sphere $x^2+y^2+z^2\le 1$ that lies in the first octant.
5. $\iiint_H z^2\,dV$, where H is the solid hemisphere $x^2+y^2+z^2\le 1$ with $z\ge 0$
6. $\iiint_H \frac{dV}{\sqrt{x^2+y^2+z^2}}$, where H is the solid hemisphere $x^2+y^2+z^2\le 1$, with $z\ge 0$
7. $\iiint_D \frac{dV}{x^2+y^2+z^2}$, where D is the solid region bounded below by the paraboloid $2z=x^2+y^2$ and above by the sphere $x^2+y^2+z^2=8$
8. $\iiint_D z(x^2+y^2)^{-1/2}\,dV$, where D is the solid bounded by the surface $2z=x^2+y^2$ and the plane $z=2$

9. Rewrite the triple integral $\int_0^1 \int_0^x \int_0^{\sqrt{xy}} f(x, y, z)\, dz\, dy\, dx$ as a triple integral in the order $dy\, dx\, dz$.

10. Express the integral

$$\int_0^1 \int_0^y f(x, y)\, dx\, dy + \int_1^4 \int_0^{(4-y)/3} f(x, y)\, dx\, dy$$

as a double integral with the order of integration reversed.

11. Find the Jacobian $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ of the change of variables $u = 2x - 3y + z$, $v = 2y - z$, $w = 2z$.

12. Find the Jacobian $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ of the change of variables $u = x^2 + y^2 + z^2$, $v = 2y^2 + z^2$, $w = 2z^2$.

13. Let $u = 2x - y$ and $v = x + 2y$. Find the image of the unit square given by $0 \le x \le 1$, $0 \le y \le 1$.

14. Find the mass of a lamina with density $\delta = r\theta$ that covers the region enclosed by the rose $r = \cos 3\theta$ for $0 \le \theta \le \frac{\pi}{6}$.

15. Suppose $u = \frac{1}{2}(x^2 + y^2)$ and $v = \frac{1}{2}(x^2 - y^2)$, with $x > 0$, $y > 0$.

   a. Find the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$.

   b. Solve for x and y in terms of u and v, and find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$.

   c. Verify that $\dfrac{\partial(u, v)}{\partial(x, y)} \dfrac{\partial(x, y)}{\partial(u, v)} = 1$.

16. **Putnam Examination Problem** The function $K(x, y)$ is positive and continuous for $0 \le x \le 1$, $0 \le y \le 1$ and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that for all $0 \le x \le 1$, we have

$$\int_0^1 f(y) K(x, y)\, dy = g(x)$$

and

$$\int_0^1 g(y) K(x, y)\, dy = f(x)$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

17. **Putnam Examination Problem** A circle of radius a is revolved through 180° about a line in its plane, distant b from the center of the circle $(b > a)$. For what value of the ratio b/a does the center of gravity of the solid thus generated lie on the surface of the solid?

18. **Putnam Examination Problem** For $f(x)$ a positive, monotone, decreasing function defined in $0 \le x \le 1$, prove that

$$\frac{\int_0^1 x[f(x)]^2\, dx}{\int_0^1 x f(x)\, dx} \le \frac{\int_0^1 [f(x)]^2\, dx}{\int_0^1 f(x)\, dx}$$

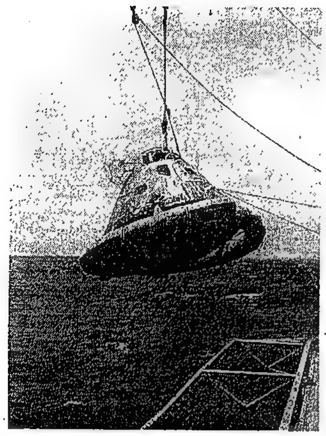
19. **Putnam Examination Problem** Show that the integral equation

$$f(x, y) = 1 + \int_0^x \int_0^y f(u, v)\, du\, dv$$

has at most one continuous solution for $0 \le x \le 1$, $0 \le y \le 1$.

Group Research Project*

Space-Capsule Design



This project is to be done in groups of three or four students. Each group will submit a single written report.

Suppose you were part of a team of engineers designing the *Apollo* space capsule. The capsule is composed of two parts:

1. A cone with a height of 4 meters and a base of radius 3 meters.
2. A reentry shield in the shape of a parabola revolved about the axis of the cone, which is attached to the cone along the edge of the base of the cone. Its vertex is a distance D below the base of the cone. Find values of the design parameters D and δ so that the capsule will float with the vertex of the cone pointing up and with the waterline 2 m below the top of the cone, in order to keep the exit port 1/3 m above water.

   You may make the following assumptions:

   a. The capsule has uniform density δ.

   b. The center of mass of the capsule should be below the center of mass of the displaced water because this will give the capsule better stability in heavy seas.

   c. A body floats in a fluid at the level at which the weight of the displaced fluid equals the weight of the body (Archimedes' principle).

THE SCIENCE OF MATHEMATICS has grown to such vast proportion that probably no living mathematician can claim to have achieved its mastery as a whole.

—A.N. Whitehead
An Introduction to Mathematics
(New York, 1911), p. 252

Your paper is not limited to the following questions but should include these concerns: Show the project director that the task is impossible; that is, there are no values of D and δ that satisfy the design specifications. However, you can solve this dilemma by incorporating a flotation collar in the shape of a torus. The collar will be made by taking hollow plastic tubing with a circular cross section of radius 1 m and wrapping it in a circular ring about the capsule, so that it fits snugly. The collar is designed to float just submerged with its top tangent to the surface of the water. Show that this flotation collar makes the capsule plus collar assembly satisfy the design specifications. Find the density δ needed to make the capsule float at the 2 meter mark. Assume that the weight of the tubing is negligible compared to the weight of the capsule, that the design parameter D is equal to 1 meter, and the density of the water is 1.

**MAA Notes* 17 (1991), "Priming the Calculus Pump: Innovations and Resources," by Marcus S. Cohen, Edward D. Gaughan, R. Arthur Knoebel, Douglas S. Kurtz, and David J. Pengelley.

13

Vector Analysis

① Green's Th. (only simply connected)
② Gauss's Th.
③ Stoke → no proof

CONTENTS

PREVIEW

In this chapter, we combine what we have learned about differentiation, integration, and vectors to study the calculus of vector functions defined on a set of points in $\mathbb{R}^2$ or $\mathbb{R}^3$. We introduce **line integrals** and **surface integrals** to study such things as fluid flow and then obtain a result called **Green's theorem** that enables line integrals to be computed in terms of ordinary double integrals. This result is extended into $\mathbb{R}^3$ to obtain **Stokes' theorem** and the **divergence theorem**, which have extensive applications in areas such as fluid dynamics and electromagnetic theory.

PERSPECTIVE

How much work is done by a variable force acting along a given curve in space? How can the amount of heat flowing across a particular surface in unit time be measured, and is the measurement similar to measuring the flow of water or electricity? We will use line integrals and surface integrals to answer these and other questions from physics and engineering mathematics.

13.1 Properties of a Vector Field: Divergence and Curl

IN THIS SECTION definition of a vector field, divergence, curl

DEFINITION OF A VECTOR FIELD

The satellite photograph in Figure 13.1 shows wind measurements over the Atlantic Ocean. Wind direction is indicated by directed line segments. This is an example of a *vector field*, in which every point in a given region of the plane or space is assigned a vector. Here is the definition of a vector field in $\mathbb{R}^3$.

Vector Field

A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns a vector to each point in its domain. A vector field with domain D in $\mathbb{R}^3$ has the form

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

where the scalar functions M, N, and P are called the **components** of $\mathbf{F}$. A **continuous** vector field $\mathbf{F}$ is one whose components M, N, and P are continuous.

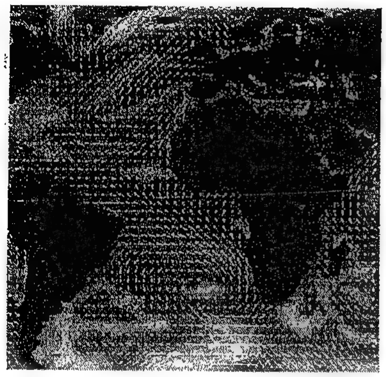


Figure 13.1 A wind-velocity map of the Atlantic Ocean

For example,

$$\mathbf{F} = 2x^2y\mathbf{i} + e^{yz}\mathbf{j} + \left(\tan\frac{x}{2}\right)\mathbf{k}$$

is a vector field with **i**-component $2x^2y$, **j**-component e^{yz}, and **k**-component $\tan\frac{x}{2}$.

A vector field in $\mathbb{R}^2$ can be thought of as a special case where there are no z-coordinates and no **k**-components. That is, a vector field in $\mathbb{R}^2$ has the form

$$\boxed{\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}}$$

To visualize a particular vector field $\mathbf{F}(x, y, z)$, it often helps to select a number of points in the domain of $\mathbf{F}$ and then draw an arrow emanating from each point $P(a, b, c)$ with the direction of $\mathbf{F}(a, b, c)$ and length representing the magnitude $\|\mathbf{F}(a, b, c)\|$. We will refer to such a representation as the **graph of F**. Here is an example involving the graph of a vector field in $\mathbb{R}^2$.

EXAMPLE 1 Graph of a vector field

Sketch the graph of the vector field $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$.

Solution

We will evaluate $\mathbf{F}$ at various points. For example,

$$\mathbf{F}(3, 4) = 4\mathbf{i} - 3\mathbf{j} \qquad \text{and} \qquad \mathbf{F}(-1, 2) = 2\mathbf{i} - (-1)\mathbf{j} = 2\mathbf{i} + \mathbf{j}$$

We can generate as many such vector values of $\mathbf{F}$ as we wish. Several are shown in Figure 13.2. ■

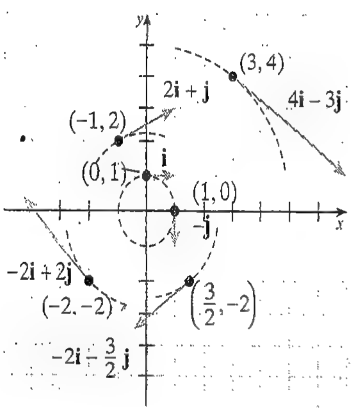


Figure 13.2 The graph of the vector field $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

The graph of a vector field often yields useful information about the properties of the field. For instance, suppose $\mathbf{F}(x, y)$ represents the velocity of a compressible fluid (like a gas) at a point (x, y) in the plane. Then $\mathbf{F}$ assigns a velocity vector to each point in the plane, and the graph of $\mathbf{F}$ provides a picture of the fluid flow. Thus, the flow in Figure 13.4a is a constant, whereas Figure 13.4b suggests a circular flow.

TECHNOLOGY NOTE

The examples in Figure 13.3 show some vector fields obtained using computer software.

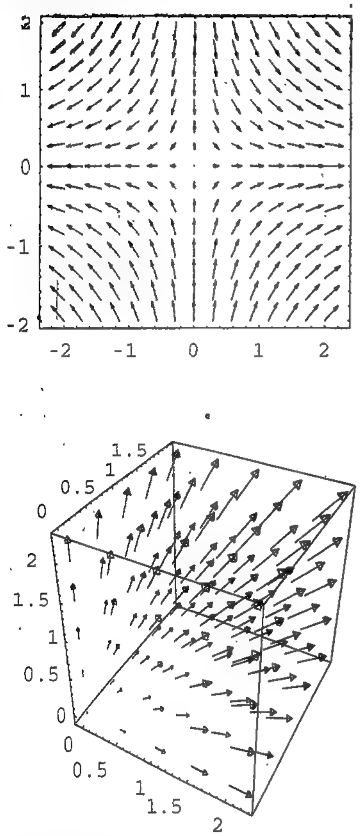
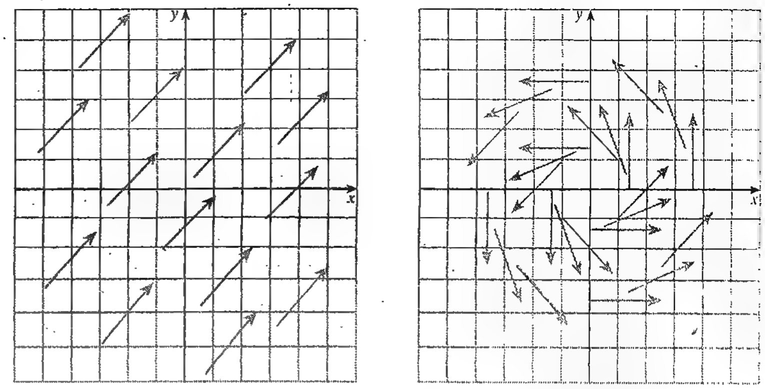


Figure 13.3 Computer generated vector fields



a. A constant fluid flow b. A circular flow

Figure 13.4 Flow diagrams

Gravitational, electrical, and magnetic vector fields play an important role in physical applications. We will discuss gravitational fields now, and electrical and magnetic fields later in this section. Accordingly, we begin with Newton's law of gravitation, which says that a point mass (particle) m at the origin exerts on a unit point mass located at the point $P(x, y, z)$ a force $\mathbf{F}(x, y, z)$ given by

$$\boxed{\mathbf{F}(x, y, z) = \frac{Gm}{x^2 + y^2 + z^2}\mathbf{u}(x, y, z)}$$

where G is a constant (the universal gravitational constant) and $\mathbf{u}$ is the unit vector extending from the point P toward the origin. The vector field $\mathbf{F}(x, y, z)$ is called the **gravitational field** of the point mass m. Because

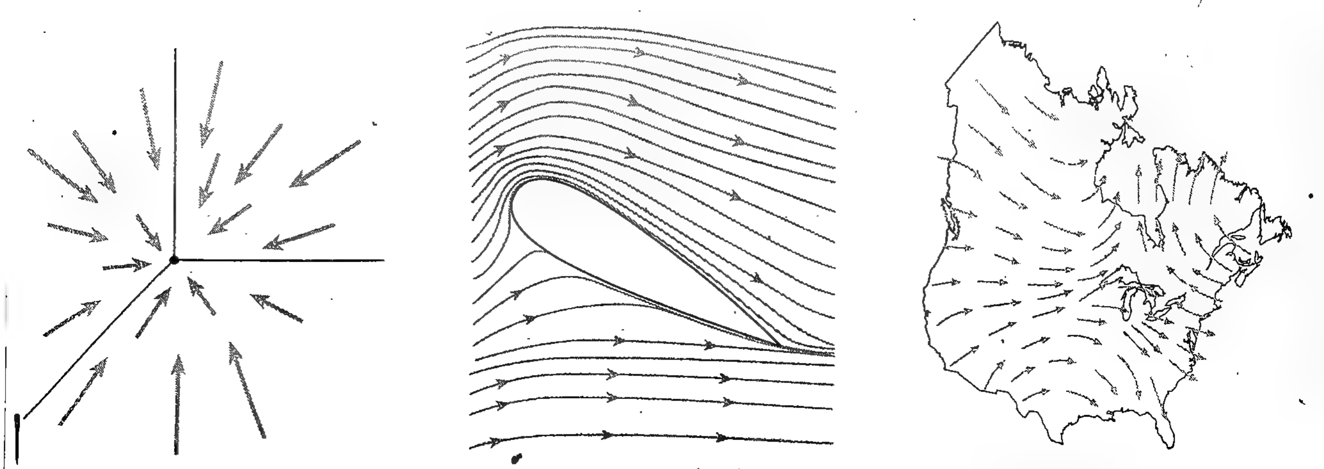
$$\mathbf{u}(x, y, z) = \frac{-1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

it follows that

$$\mathbf{F}(x, y, z) = \frac{-Gm}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

Always pts towards origin & has same mag. for same r

Note that the gravitational field $\mathbf{F}$ always points toward the origin and has the same magnitude for any point m located $r = \sqrt{x^2 + y^2 + z^2}$ units from the origin. Such a vector field is called a **central force field**. This force field is shown in Figure 13.5a. Other physical vector fields are shown in Figures 13.5b and 13.5c.



a. A central force field b. Air flow vector field c. Wind velocity on a map

DIVERGENCE

Divergence and *curl* are two operations on vector fields that originated in connection with the study of fluid flow. Divergence may be defined as follows.

Divergence

WARNING The divergence of a vector field is a scalar function.

> The **divergence** of a vector field
> $$\mathbf{V}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$$
> is denoted by div $\mathbf{V}$ and is given by
> $$\text{div}\,\mathbf{V} = \frac{\partial u}{\partial x}(x, y, z) + \frac{\partial v}{\partial y}(x, y, z) + \frac{\partial w}{\partial z}(x, y, z)$$

EXAMPLE 2 Divergence of a vector field

Find the divergence of each of the following vector fields.

a. $\mathbf{F}(x, y) = x^2y\,\mathbf{i} + xy^3\,\mathbf{j}$
b. $\mathbf{G}(x, y, z) = x\,\mathbf{i} + y^3z^2\,\mathbf{j} + xz^3\,\mathbf{k}$

Solution

a. $\text{div}\,\mathbf{F} = \dfrac{\partial}{\partial x}(x^2y) + \dfrac{\partial}{\partial y}(xy^3) = 2xy + 3xy^2$

b. $\text{div}\,\mathbf{G} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y^3z^2) + \dfrac{\partial}{\partial z}(xz^3) = 1 + 3y^2z^2 + 3xz^2$ ■

Suppose the vector field

$$\mathbf{V}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$$

represents the velocity of a fluid with density $\delta\,(x, y, z)$ at a point (x, y, z) in a certain region R in $\mathbb{R}^3$. Then the vector field $\delta\mathbf{V}$ is called the **flux density** and is denoted by $\mathbf{D}$. We can think of $\mathbf{D} = \delta\mathbf{V}$ as measuring the "mass flow" of the liquid.

Assuming there are no external processes acting on the fluid that would tend to create or destroy fluid, it can be shown that div $\mathbf{D}$ gives the negative of the time rate change of the density, that is,

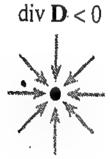
$$\boxed{\text{div}\,\mathbf{D} = -\frac{\partial \delta}{\partial t}}$$

This is often referred to as the **continuity equation** of fluid dynamics. (A derivation is given in Section 13.7.) When div $\mathbf{D} = 0$, $\mathbf{D}$ is said to be **incompressible**. If div $\mathbf{D} > 0$ at a point (x_0, y_0, z_0), the point is called a **source**; if div $\mathbf{D} < 0$, the point is called a **sink** (see Figure 13.6). The terms *sink*, *source*, and *incompressible* apply to any vector field $\mathbf{F}$ and are not reserved only for fluid applications.

div $\mathbf{D} > 0$

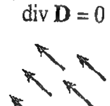
Fluid flows from a *source point.*

div $\mathbf{D} < 0$



Fluid flows toward a *sink point.*

div $\mathbf{D} = 0$



Fluid is *incompressible.*

Figure 13.6 Flow of a fluid across a plane region, $\mathbf{D}$

A useful way to think of the divergence div $\mathbf{V}$ is in terms of the **del operator** defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Recall (from Section 11.6) that applying the del operator to the differentiable function $f(x, y, z)$ produces the **gradient field**

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Similarly, by taking the dot product of the operator ∇ with the vector field $\mathbf{V} = u(x, y, z)\,\mathbf{i} + v(x, y, z)\,\mathbf{j} + w(x, y, z)\,\mathbf{k}$, we obtain the divergence

$$\begin{aligned}\nabla\cdot\mathbf{V} &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\\ &= \frac{\partial}{\partial x}(u)+\frac{\partial}{\partial y}(v)+\frac{\partial}{\partial z}(w)\\ &= \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\\ &= \operatorname{div}\mathbf{V}\end{aligned}$$

CURL

The del operator may also be used to describe another derivative operation for vector fields, called the *curl*.

Curl

The **curl** of a vector field

$$\mathbf{V}(x,y,z)=u(x,y,z)\mathbf{i}+v(x,y,z)\mathbf{j}+w(x,y,z)\mathbf{k}$$

is denoted by curl **V** and is defined by

$$\operatorname{curl}\mathbf{V}=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\mathbf{i}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\mathbf{k}$$

Note that

$$\begin{aligned}\operatorname{curl}\mathbf{V} &= \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\mathbf{i}-\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)\mathbf{j}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\mathbf{k}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{vmatrix}\begin{array}{l}\leftarrow \text{Standard basis vectors}\\ \leftarrow \nabla\\ \leftarrow \mathbf{V}\end{array}\\ &= \nabla\times\mathbf{V}\end{aligned}$$

The determinant form of curl **V** is a convenient device for remembering the definition and is helpful in organizing computations.

WARNING Notice that div **V** is a scalar, and curl **V** is a vector.

Del Operator Forms for Divergence and Curl

Consider a vector field

$$\mathbf{V}(x,y,z)=u(x,y,z)\mathbf{i}+v(x,y,z)\mathbf{j}+w(x,y,z)\mathbf{k}$$

The divergence and curl of **V** are given by

$$\operatorname{div}\mathbf{V}=\nabla\cdot\mathbf{V}\quad\text{and}\quad\operatorname{curl}\mathbf{V}=\nabla\times\mathbf{V}$$

EXAMPLE 3 Curl of a vector field

Find the curl of each of the following vector fields:

$$\mathbf{F}=x^2yz\mathbf{i}+xy^2z\mathbf{j}+xyz^2\mathbf{k}\quad\text{and}\quad\mathbf{G}=(x\cos y)\mathbf{i}+xy^2\mathbf{j}$$

Solution

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}xyz^2 - \frac{\partial}{\partial z}xy^2z\right)\mathbf{i} - \left(\frac{\partial}{\partial x}xyz^2 - \frac{\partial}{\partial z}x^2yz\right)\mathbf{j} + \left(\frac{\partial}{\partial x}xy^2z - \frac{\partial}{\partial y}x^2yz\right)\mathbf{k}$$

$$= (xz^2 - xy^2)\mathbf{i} + (x^2y - yz^2)\mathbf{j} + (y^2z - x^2z)\mathbf{k}$$

$$\operatorname{curl}\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\cos y & xy^2 & 0 \end{vmatrix}$$

$$= \left[0 - \frac{\partial}{\partial z}xy^2\right]\mathbf{i} - \left[0 - \frac{\partial}{\partial z}(x\cos y)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}xy^2 - \frac{\partial}{\partial y}(x\cos y)\right]\mathbf{k}$$

$$= (y^2 + x\sin y)\mathbf{k}$$

■

EXAMPLE 4 A constant vector field has divergence and curl zero

Let $\mathbf{F}$ be a constant vector field. Show that $\operatorname{div}\mathbf{F} = 0$ and $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

Solution

Let $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ for constants a, b, and c. Then

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(a) + \frac{\partial}{\partial y}(b) + \frac{\partial}{\partial z}(c) = 0$$

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a & b & c \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

■

WARNING Example 4 shows that the divergence and curl of a constant vector field are zero, but this does not mean that if $\operatorname{div}\mathbf{F} = 0$ and $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ must be a constant. For instance, the nonconstant vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$$

has both $\operatorname{div}\mathbf{F} = 0$ and $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

TECHNOLOGY NOTE

Derive, *Maple*, MATLAB, and *Mathematica* will carry out most vector operations. For Example 2, we define $\mathbf{G}(x, y, z) = x\mathbf{i} + y^3z^2\mathbf{j} + xz^3\mathbf{k}$, and use one of these programs:

Div[x, $y^3\,z^2$, $x\,z^3$] simplifies to $3xz^2 + 3y^2z^2 + 1$

You may need to enter a zero, as with $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^3\mathbf{j}$, as shown here:

Div[$x^2\,y$, $x\,y^3$, 0] simplifies to $x(3y^2 + 2y)$

Finally, consider the vector field $\mathbf{F}$ from Example 3, $\mathbf{F} = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$. We can find the curl:

Curl[$x^2\,yz$, xy^2z, xyz^2] simplifies to $[x(z^2 - y^2), y(x^2 - z^2), z(y^2 - x^2)]$

Combinations of the gradient, divergence, and curl appear in a variety of applications. In particular, note that if f is a differentiable scalar function, its gradient ∇f is a vector field, and we can compute

$$\begin{aligned} \operatorname{div} \nabla f &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \nabla \cdot \nabla f \end{aligned}$$

In the following box, we introduce some special notation and terminology for this operation.

The Laplacian Operator

The Laplacian is named for the French mathematician Pierre Laplace (1749–1827). See the Historical Quest, Section 12.3, Problem 22.

Let $f(x, y, z)$ define a function with continuous first and second partial derivatives. Then the **Laplacian of f** is

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = f_{xx} + f_{yy} + f_{zz}$$

The equation $\nabla^2 f = 0$ is called **Laplace's equation**, and a function that satisfies such an equation in a region D is said to be **harmonic** in D.

EXAMPLE 5 Showing a function is harmonic

Show that $f(x, y) = e^x \cos y$ is harmonic.

Solution

$$f_x(x, y) = e^x \cos y \quad \text{and} \quad f_{xx}(x, y) = e^x \cos y$$
$$f_y(x, y) = -e^x \sin y \quad \text{and} \quad f_{yy}(x, y) = -e^x \cos y$$

The Laplacian of f is given by

$$\begin{aligned} \nabla^2 f(x, y) &= f_{xx}(x, y) + f_{yy}(x, y) \qquad f_{zz} = 0 \\ &= e^x \cos y - e^x \cos y = 0 \end{aligned}$$

Thus, f is harmonic. ■

TECHNOLOGY NOTE

Derive, *Maple*, MATLAB, and *Mathematica* will find the Laplacian of a given function. For Example 5, we obtain

LAPLACIAN (e^x cos(y), [x, y]) which gives 0

If you have access to this technology, verify that

LAPLACIAN ($x^2 y^3 z$, [x, y, z]) yields $6x^2yz + 2y^3z$

In many ways, the study of electricity and magnetism is analogous to that of fluid dynamics, and the curl and divergence play an important role in this study. In electromagnetic theory, it is often convenient to regard interaction between electrical charges as forces somewhat like the gravitational force between masses and then to seek quantitative measure of these forces.

One of the great scientific achievements of the nineteenth century was the discovery of the laws of electromagnetism by the English scientist James Clerk Maxwell (see Historical Quest, Section 13.7, Problem 32). These laws have an

elegant expression in terms of the divergence and curl. It is known empirically that the force acting on a charge due to an electromagnetic field depends on the position, velocity, and amount of the particular charge, and not on the number of other charges that may be present or how those other charges are moving. Suppose a charge is located at the point (x, y, z) at time t, and consider the electric intensity field $\mathbf{E}(x, y, z, t)$ and the magnetic intensity field $\mathbf{H}(x, y, z, t)$. Then the behavior of the resulting electromagnetic field is determined by

$$\operatorname{div} \mathbf{E} = \frac{Q}{\epsilon} \qquad \operatorname{div}(\mu \mathbf{H}) = 0$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t}(\mu \mathbf{H}) \qquad c^2(\operatorname{curl} \mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} + \frac{\mathbf{J}}{\epsilon}$$

where Q is the *electric charge density* (charge per unit volume), $\mathbf{J}$ is the *electric current density* (rate at which the charge flows through a unit area per second), $\mathbf{B}$ is the magnetic flux density, c is the speed of light, and μ and ϵ are constants called the *permeability* and *permittivity*, respectively. Working with these equations and terms is beyond the scope of this course, but if you are interested there are many references you can consult. One of the best (despite being almost 40 years old) is the classic *Feynman Lectures in Physics* (Reading, Mass.: Addison-Wesley, 1963), by Nobel laureate Richard Feynman, Robert Leighton, and Matthew Sands.

HW 8 Pg 135 M1 Guidelines

13.1 PROBLEM SET

[Q 3, 5, 7, 9, 11, 13, 15, 17, 19, 46–55

A 1. **Exploration Problem** Discuss the del operator and its use in computing the divergence and curl.

2. **Exploration Problem** Discuss the difference between a vector valued function and a vector field.

In Problems 3–6, find div **F** *and* curl **F** *for the given vector function.*

3. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j} + z^3\mathbf{k}$

4. $\mathbf{F}(x, y) = \mathbf{i} + (x^2 + y^2)\mathbf{j}$

5. $\mathbf{F}(x, y, z) = 2y\mathbf{j}$

6. $\mathbf{F}(x, y, z) = z\mathbf{i} - \mathbf{j} + 2y\mathbf{k}$

In Problems 7–12, find div **F** *and* curl **F** *for each vector field* **F** *at the given point.*

7. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ at $(2, -1, 3)$
8. $\mathbf{F}(x, y, z) = xz\mathbf{i} + y^2z\mathbf{j} + xz\mathbf{k}$ at $(1, -1, 2)$
9. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ at $(1, 2, 3)$
10. $\mathbf{F}(x, y, z) = (\cos y)\mathbf{i} + (\sin y)\mathbf{j} + \mathbf{k}$ at $(\frac{\pi}{4}, \pi, 0)$
11. $\mathbf{F}(x, y, z) = e^{-xy}\mathbf{i} + e^{xz}\mathbf{j} + e^{yz}\mathbf{k}$ at $(3, 2, 0)$
12. $\mathbf{F}(x, y, z) = (e^{-x}\sin y)\mathbf{i} + (e^{-x}\cos y)\mathbf{j} + \mathbf{k}$ at $(1, 3, -2)$

Find div **F** and curl **F** *for each vector field* **F** *given in Problems 13–28.*

13. $\mathbf{F} = (\sin x)\mathbf{i} + (\cos y)\mathbf{j}$
14. $\mathbf{F} = (-\cos x)\mathbf{i} + (\sin y)\mathbf{j}$
15. $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$
16. $\mathbf{F} = -x\mathbf{i} + y\mathbf{j}$
17. $\mathbf{F} = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$
18. $\mathbf{F} = x^2\mathbf{i} - y^2\mathbf{j}$
19. $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + c\mathbf{k}$ for constants a, b, and c
20. $\mathbf{F} = (e^x \sin y)\mathbf{i} + (e^x \cos y)\mathbf{j} + \mathbf{k}$
21. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
22. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$
23. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
24. $\mathbf{F} = 2xz\mathbf{i} + 2yz^2\mathbf{j} - \mathbf{k}$
25. $\mathbf{F} = xyz\mathbf{i} + x^2y^2z^2\mathbf{j} + y^2z^3\mathbf{k}$
26. $\mathbf{F} = (\ln z)\mathbf{i} + e^{xy}\mathbf{j} + \tan^{-1}\left(\dfrac{x}{z}\right)\mathbf{k}$
27. $\mathbf{F} = (z^2e^{-x})\mathbf{i} + (y^3 \ln z)\mathbf{j} + (xe^{-y})\mathbf{k}$
28. $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

Determine whether each scalar function in Problems 29–32 is harmonic.

29. $u(x, y, z) = e^{-x}(\cos y - \sin y)$
30. $v(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$
31. $w(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
32. $r(x, y, z) = xyz$
33. Show that the vector field $\mathbf{B} = y^2z\mathbf{i} + xz^3\mathbf{j} + y^2x^2\mathbf{k}$ is incompressible (that is, $\operatorname{div}\mathbf{B} = 0$).
34. Find div **F**, given that $\mathbf{F} = \nabla f$, where $f(x, y, z) = xy^3z^2$.
35. Find div **F**, given that $\mathbf{F} = \nabla f$, where $f(x, y, z) = x^2yz^3$.
36. If $\mathbf{F}(x, y, z) = 2\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$, find $\operatorname{curl}(\mathbf{F} \times \mathbf{G})$.
37. If $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$, find $\operatorname{curl}(\mathbf{F} \times \mathbf{G})$.
38. If $\mathbf{F}(x, y, z) = 2\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$, find $\operatorname{div}(\mathbf{F} \times \mathbf{G})$.
39. If $F(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$, find $\operatorname{div}(\mathbf{F} \times \mathbf{G})$.

B 40. Let **A** be a constant vector and let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\operatorname{div}(\mathbf{A} \times \mathbf{R}) = 0$.
41. Let **A** be a constant vector and let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\operatorname{curl}(\mathbf{A} \times \mathbf{R}) = 2\mathbf{A}$.
42. If $\mathbf{F}(x, y) = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$, show that $\operatorname{curl}\mathbf{F} = \mathbf{0}$ if and only if $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$.

CW copy See 189 110

43. Consider a rigid body that is rotating about the z-axis (counterclockwise from above) with constant angular velocity $\boldsymbol{\omega} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. If P is a point in the body located at $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the velocity at P is given by the vector field $\mathbf{V} = \boldsymbol{\omega} \times \mathbf{R}$.
    a. Express $\mathbf{V}$ in terms of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.
    b. Find div $\mathbf{V}$ and curl $\mathbf{V}$.

44. **Exploration Problem** If $\mathbf{F} = \langle f, g, h \rangle$ is an arbitrary vector field whose components are twice differentiable, what can be said about curl (curl $\mathbf{F}$)?

45. Which (if any) of the following is the same as div $(\mathbf{F} \times \mathbf{G})$ for all vector fields $\mathbf{F}$ and $\mathbf{G}$?
    I. $(\operatorname{div}\mathbf{F})(\operatorname{div}\mathbf{G})$
    II. $(\operatorname{curl}\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\operatorname{curl}\mathbf{G})$
    III. $\mathbf{F}(\operatorname{div}\mathbf{G}) + (\operatorname{div}\mathbf{F})\mathbf{G}$
    IV. $(\operatorname{curl}\mathbf{F}) \cdot \mathbf{G} + \mathbf{F} \cdot (\operatorname{curl}\mathbf{G})$

In Problems 46–55 prove the given property for the vector fields $\mathbf{F}$ *and* $\mathbf{G}$, *scalar* c, *and scalar functions* f *and* g. *Assume that all required partial derivatives exist and are continuous.*

46. div $(c\mathbf{F}) = c$ div $\mathbf{F}$ Pg 111 CW
47. div $(\mathbf{F} + \mathbf{G}) =$ div $\mathbf{F}$ + div $\mathbf{G}$ Pg 111 CW
48. curl$(\mathbf{F} + \mathbf{G}) =$ curl $\mathbf{F}$ + curl $\mathbf{G}$
49. curl $(c\,\mathbf{F}) = c$ curl $\mathbf{F}$ Pg 112 CW
50. curl $(f\mathbf{F}) = f$ curl $\mathbf{F} + (\nabla f \times \mathbf{F})$ Pg 118 CW
51. div $(f\mathbf{F}) = f$ div $\mathbf{F} + (\nabla f \cdot \mathbf{F})$ Pg 119 CW
52. curl $(\nabla f + \text{curl}\,\mathbf{F}) =$ curl (∇f) + curl (curl $\mathbf{F}$) Pg 120 CW
53. div $(f \nabla g) = f$ div $\nabla g + \nabla f \cdot \nabla g$ Pg 121 CW
54. The curl of the gradient of a function is always $\mathbf{0}$. That is, $\nabla \times (\nabla f) = \mathbf{0}$. Pg 116 CW
55. The divergence of the curl of a vector field is 0. That is, div (curl $\mathbf{F}$) = 0. Pg 117 CW

C *In Problems 56–60,* $\mathbf{R} = \langle x, y, z \rangle$, *and* $r = \|\mathbf{R}\| = \sqrt{x^2 + y^2 + z^2}$. *In each case, verify the given identity or answer the question.*

div $\vec{R} = 1+1+1 = 3.$

56. curl $\mathbf{R} = \mathbf{0}$; what is div $\mathbf{R}$?
57. $\operatorname{div}\left(\dfrac{1}{r^3}\mathbf{R}\right) = 0$ CW Pg 115
58. $\operatorname{curl}\left(\dfrac{1}{r^3}\mathbf{R}\right) = \mathbf{0}$ Pg 114 CW
59. $\operatorname{div}(r\mathbf{R}) = 4r$ CW Pg 114
60. $\operatorname{div}(\nabla r) = \dfrac{2}{r}$ Pg 118

61. **Exploration Problem** State and prove an identity for $\operatorname{div}(\nabla(fg))$, where f and g are differentiable scalar functions of x, y, and z.

62. **Counterexample Problem** Let $\mathbf{F} = \langle x^2y, yz^2, zy^2 \rangle$. Either find a vector field $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl}\mathbf{G}$, or show that no such $\mathbf{G}$ exists.

(57) $\frac{1}{r^3} \cdot 3 + \left(-\frac{3}{2}(x^2+y^2+z^2)^{-5/2} \cdot (2x\hat{i} + 2y\hat{j} + 2z\hat{k})\right)$

$= \frac{3}{(x^2+y^2+z^2)^{3/2}} - 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) = 0$

13.2 Line Integrals

IN THIS SECTION definition of a line integral; line integrals with respect to x, y, and z; line integrals of vector fields; applications of line integrals: mass and work

A line integral is an integral whose integrand is evaluated at points along a curve in $\mathbb{R}^2$ or in $\mathbb{R}^3$. We will introduce line integrals in this section and show how they can be used for a variety of purposes in mathematics and physics.

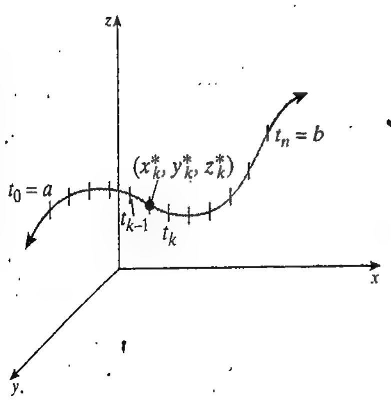


Figure 13.7 The curve C partitioned into subarcs

DEFINITION OF A LINE INTEGRAL

Def. Let C be a smooth curve, with parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \le t \le b$, that lies within the domain of a function $f(x, y, z)$. We say that C is **orientable** if it is possible to describe direction along the curve for increasing t.

cont. differentiable

To define a line integral, we begin by partitioning C into n subarcs, the kth of which has length Δs_k. Let (x_k^*, y_k^*, z_k^*) be a point chosen arbitrarily from the kth subarc (see Figure 13.7). Form the Riemann sum

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k$$

and let $\|\Delta s\|$ denote the largest subarc length in the partition. Then, if the limit

$$\lim_{\|\Delta s\| \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k$$

exists, we call this limit the *line integral* of f over C and denote it by $\displaystyle\int_C f(x, y, z)\, ds$.

Line Integral over a Smooth Curve

If $f(x, y, z)$ is defined on the smooth curve C with parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, then the **line integral** of f over C is given by

$$\int_C f(x, y, z)\, ds = \lim_{\|\Delta s\| \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k$$

provided that this limit exists. If C is a closed curve, we indicate the line integral of f around C by $\boxed{\oint_C f\, ds.}$

It can be shown that the limit that defines the line integral $\int_C f\, ds$ always exists if f is continuous at each point of C. Also, since the curve C is smooth, the component functions $x(t)$, $y(t)$, and $z(t)$ will all be continuously differentiable. Thus, we have

$$\boxed{ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt}$$

so the line integral can be written entirely in terms of t:

C curve in $\mathbb{R}^3$

$$\boxed{\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt}$$

Similarly, if $f(x, y)$ is a function of only two variables and C is a curve in the plane, then

C curve in $\mathbb{R}^2$

$$\boxed{\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$$

EXAMPLE 1 Evaluating a line integral in three variables

Evaluate the line integral $\int_C x^2 z\, ds$, where C is the helix $x = \cos t$, $y = 2t$, $z = \sin t$, for $0 \le t \le \pi$.

Solution

Since $x'(t) = -\sin t$, $y'(t) = 2$, $z'(t) = \cos t$, the line integral is

$$\begin{aligned}
\int_C x^2 z\, ds &= \int_0^\pi [x(t)]^2 z(t)\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt \\
&= \int_0^\pi (\cos t)^2 (\sin t)\sqrt{(-\sin t)^2 + (2)^2 + (\cos t)^2}\, dt \\
&= \int_0^\pi \sqrt{5}\cos^2 t \sin t\, dt \\
&= \frac{-\sqrt{5}}{3}\cos^3 t \Big|_0^\pi \\
&= \frac{2\sqrt{5}}{3}
\end{aligned}$$

■

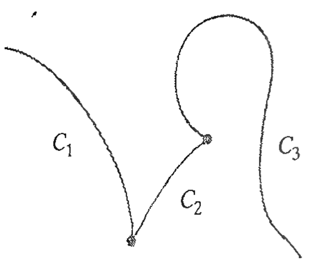


Figure 13.8 Piecewise smooth curve

The definition of a line integral can be extended to curves that are *piecewise smooth* in the sense that they are the union of a finite number of smooth curves with only endpoints in common, as shown in Figure 13.8.

In particular, if C is comprised of a number of smooth subarcs $C_1, C_2, \ldots, C_n$, then

$$\int_C f(x, y, z)\, ds = \int_{C_1} f(x, y, z)\, ds + \int_{C_2} f(x, y, z)\, ds + \cdots + \int_{C_n} f(x, y, z)\, ds$$

This definition of line integration is illustrated in the following example.

EXAMPLE 2 Evaluating a line integral over a union of curves

Evaluate the line integral $\int_C xy\,ds$, where C consists of the line segment C_1 from $(-3, 3)$ to $(0, 0)$, followed by the portion of the curve C_2: $16y = x^4$ between $(0, 0)$ and $(2, 1)$.

Solution

The curve C is shown in Figure 13.9. The segment C_1 is part of the line $y = -x$ and can be parameterized by the equations $x = t$, $y = -t$ over the interval $-3 \le t \le 0$. On this curve, we have

$$x'(t) = 1, \quad y'(t) = -1, \quad \text{so} \quad ds = \sqrt{(1)^2 + (-1)^2}\,dt = \sqrt{2}\,dt$$

and the line integral of $f(x, y) = xy$ over C_1 is

$$\int_{C_1} xy\,ds = \int_{-3}^{0} (t)(-t)\sqrt{2}\,dt = \int_{-3}^{0} -\sqrt{2}\,t^2\,dt = -9\sqrt{2}$$

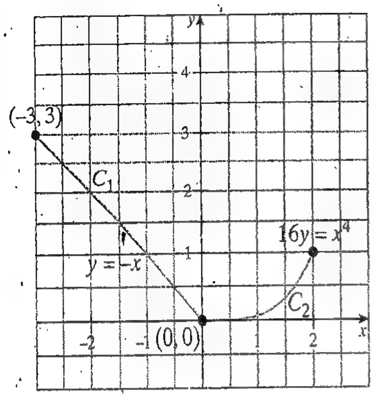


Figure 13.9 The curve C in Example 2

The curve C_2, $16y = x^4$, can be parameterized by the equations $x = 2t$, $y = \frac{1}{16}x^4 = \frac{1}{16}(2t)^4 = t^4$ for $0 \le t \le 1$. We find that

$$x'(t) = 2, \quad y'(t) = 4t^3, \quad \text{so} \quad ds = \sqrt{(2)^2 + (4t^3)^2}\,dt$$

and the line integral over C_2 is

$$\int_{C_2} xy\,ds = \int_0^1 (2t)(t^4)\sqrt{4 + 16t^6}\,dt = \frac{1}{9}(1 + 4t^6)^{3/2}\Big|_0^1 = \frac{1}{9}(5\sqrt{5} - 1)$$

Thus, the line integral over C is given by the sum

$$\int_C xy\,ds = \int_{C_1} xy\,ds + \int_{C_2} xy\,ds = -9\sqrt{2} + \frac{1}{9}(5\sqrt{5} - 1)$$

■

THEOREM 13.1 Properties of line integrals

Let f be a given scalar function defined on a piecewise smooth, orientable curve C. Then, for any constant k,

Constant multiple rule: $\displaystyle \int_C kf\,ds = k\int_C f\,ds$

Sum rule: $\displaystyle \int_C (f_1 + f_2)\,ds = \int_C f_1\,ds + \int_C f_2\,ds$

where f_1 and f_2 are scalar functions defined on C.

Opposite direction rule: $\displaystyle \boxed{\int_{-C} f\,ds = -\int_C f\,ds}$

where $-C$ denotes the curve C traversed in the opposite direction.

Subdivision rule: $\displaystyle \int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds$

where C is the union of smooth orientable subarcs $C = C_1 \cup C_2 \cup \cdots \cup C_n$ with only endpoints in common.

Proof The proof follows directly from the properties of limits and the definition of a line integral. □

LINE INTEGRALS WITH RESPECT TO x, y, AND z

If Δs is replaced by Δx in the discussion leading to the definition of the line integral $\int_C f(x, y, z)\, ds$, we obtain a definition for the line integral $\int_C f(x, y, z)\, dx$. Since $x = x(t)$ is differentiable, we have $dx = x'(t)\, dt$ and the line integral of f with respect to x can be evaluated as follows:

$$\boxed{\int_C f(x, y, z)\, dx = \int_a^b f[x(t), y(t), z(t)]x'(t)\, dt}$$

Similarly, if g and h are continuous on C, then

$$\boxed{\begin{aligned}\int_C g\, dy &= \int_a^b g[x(t), y(t), z(t)]y'(t)\, dt\\ \int_C h\, dz &= \int_a^b h[x(t), y(t), z(t)]z'(t)\, dt\end{aligned}}$$

By combining the line integrals with respect to the coordinate variables x, y, and z, we obtain a line integral of the form

$$\boxed{\int_C [f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz]}$$

EXAMPLE 3 Evaluating a line integral with respect to coordinate variables

Evaluate the line integral

$$\int_C [y\, dx - z\, dy + x\, dz]$$

where C is the curve with parametric equations $x = t^2$, $y = e^{-t}$, $z = e^t$ for $0 \le t \le 1$.

Solution

Since $x'(t) = 2t$, $y'(t) = -e^{-t}$, and $z'(t) = e^t$, we have

$$\begin{aligned}\int_C [y\, dx - z\, dy + x\, dz] &= \int_0^1 e^{-t}(2t\, dt) - e^t(-e^{-t}\, dt) + t^2(e^t\, dt)]\\ &= \int_0^1 [2te^{-t} + 1 + t^2e^t]\, dt\\ &= \left[-2e^{-t}(t + 1) + t + e^t(t^2 - 2t + 2)\right]\Big|_0^1\\ &= [-4e^{-1} + 1 + e(1 - 2 + 2)] - [-2 + 0 + 2]\\ &= e - 4e^{-1} + 1\end{aligned}$$

■

LINE INTEGRALS OF VECTOR FIELDS

We will now discuss what it means to compute the *line integral of a vector field.*

Line Integral of a Vector Field

Let $\mathbf{F}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$ be a vector field, and let C be a piecewise smooth orientable curve with parametric representation

$$\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{for } a \leq t \leq b$$

Using $d\mathbf{R} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, we define the **line integral of F along C** by

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C (u\,dx + v\,dy + w\,dz) = \int_C \mathbf{F}[\mathbf{R}(t)] \cdot \mathbf{R}'(t)\,dt$$

$$= \int_a^b \left[u[x(t), y(t), z(t)]\frac{dx}{dt} + v[x(t), y(t), z(t)]\frac{dy}{dt} + w[x(t), y(t), z(t)]\frac{dz}{dt} \right] dt$$

EXAMPLE 4 Evaluating a line integral of a vector field

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = (y^2 - z^2)\mathbf{i} + (2yz)\mathbf{j} - x^2\mathbf{k}$ and C is the curve defined parametrically by $x = t^2$, $y = 2t$, and $z = t$ for $0 \leq t \leq 1$.

Solution

Rewrite **F** using the parameter t:

$$\mathbf{F} = [(2t)^2 - (t)^2]\mathbf{i} + [2(2t)(t)]\mathbf{j} - [(t^2)^2]\mathbf{k} = 3t^2\mathbf{i} + 4t^2\mathbf{j} - t^4\mathbf{k}.$$

Because $\mathbf{R}(t) = t^2\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$, we have $d\mathbf{R} = (2t\,dt)\mathbf{i} + (2\,dt)\mathbf{j} + dt\mathbf{k}$, so

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{R} &= (3t^2)(2t\,dt) + (4t^2)(2\,dt) + (-t^4)(dt) \\ &= (6t^3 + 8t^2 - t^4)\,dt \end{aligned}$$

Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 (6t^3 + 8t^2 - t^4)\,dt = \left[\frac{3}{2}t^4 + \frac{8}{3}t^3 - \frac{1}{5}t^5\right]\Big|_0^1 = \frac{119}{30}$$ ■

A line integral over a curve C does not depend on the parameterization used for C. This property of line integrals is illustrated in the following example.

EXAMPLE 5 The value of a line integral is independent of parameterization

Let $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ and let C be the top half of the circle $x^2 + y^2 = 4$ traversed counterclockwise from $(2, 0)$ to $(-2, 0)$, as shown in Figure 13.10. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ for each of the following parameterizations of the curve C.

a. $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \leq \theta \leq \pi$

b. $x = -t$, $y = \sqrt{4 - t^2}$, $-2 \leq t \leq 2$

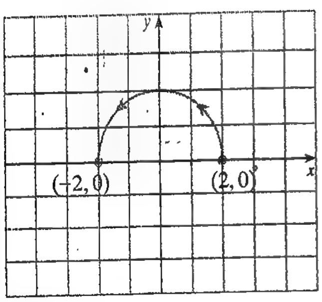


Figure 13.10 Graph of the semicircle C

Solution

a. With the parameterization $x = 2\cos\theta$, $y = 2\sin\theta$, we find that $x'(\theta) = -2\sin\theta$, $y'(\theta) = 2\cos\theta$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{R} &= \int_C [y\,dx + x\,dy] \\ &= \int_0^\pi [(2\sin\theta)(-2\sin\theta) + (2\cos\theta)(2\cos\theta)]\,d\theta \\ &= \int_0^\pi 4(\cos^2\theta - \sin^2\theta)\,d\theta \\ &= \int_0^\pi 4\cos 2\theta\,d\theta \\ &= 0 \end{aligned}$$

b. For the parameterization $x = -t$, $y = \sqrt{4-t^2}$, we have $x'(t) = -1$, $y'(t) = \dfrac{-t}{\sqrt{4-t^2}}$. We find that

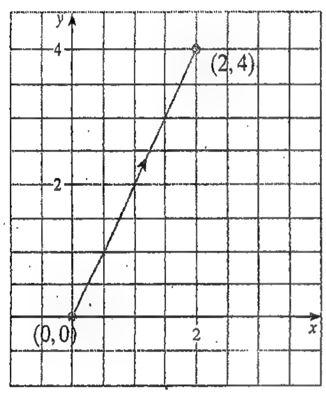
$$\begin{aligned}\int_C [y\,dx + x\,dy] &= \int_{-2}^{2}\left[\sqrt{4-t^2}(-1) + (-t)\left(\frac{-t}{\sqrt{4-t^2}}\right)\right] dt\\ &= \int_{-2}^{2} \frac{-4+2t^2}{\sqrt{4-t^2}}\,dt\\ &= \left[-t\sqrt{4-t^2}\right]\Big|_{-2}^{2}\\ &= 0\end{aligned}$$

This is the same result that was obtained using the parameterization in part **a.** ■

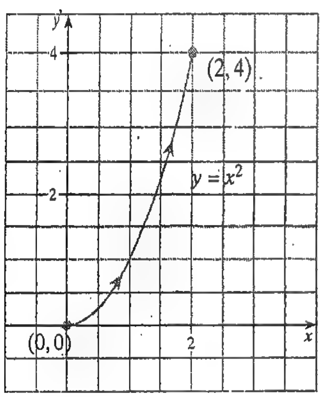
EXAMPLE 6 Evaluating line integrals along different paths

Let $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j}$ and evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{R}$ between the points (0, 0) and (2, 4) along the following paths:.

a. the line segment connecting the points
b. the parabolic arc $y = x^2$ connecting the points



a. The line segment path



b. The parabolic path

Figure 13.11 A line integral along different paths

Solution

The two paths we are considering are shown in Figure 13.11.

a. The line joining the given points has equation $y = 2x$, which may be parameterized by setting $x = t$, $y = 2t$ for $0 \le t \le 2$. Thus,

$$\mathbf{R}(t) = t\mathbf{i} + 2t\mathbf{j} \qquad \text{so that} \quad d\mathbf{R} = dt\,\mathbf{i} + 2\,dt\,\mathbf{j}$$

In terms of t, we find $\mathbf{F} = 4t^3\mathbf{i} + 2t^3\mathbf{j}$ and

$$\mathbf{F}\cdot d\mathbf{R} = 4t^3\,dt + 4t^3\,dt = 8t^3\,dt$$

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^2 8t^3\,dt = \left[2t^4\right]\Big|_0^2 = 32$$

b. The parabola $y = x^2$ can be parameterized by setting $x = t$, $y = t^2$ for $0 \le t \le 2$. Thus,

$$\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} \qquad \text{so that} \quad d\mathbf{R} = dt\,\mathbf{i} + 2t\,dt\,\mathbf{j}$$

In terms of t,

$$\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} = (t)(t^2)^2\mathbf{i} + (t)^2(t^2)\mathbf{j} = t^5\mathbf{i} + t^4\mathbf{j}$$

and

$$\mathbf{F}\cdot d\mathbf{R} = t^5\,dt + 2t^5\,dt = 3t^5\,dt$$

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^2 3t^5\,dt = \left[\frac{1}{2}t^6\right]\Big|_0^2 = 32 \quad ■$$

In Example 6 we see that the value of the line integral is the same for both paths. Indeed, it can be shown that for the vector field $\mathbf{F} = xy^2\,\mathbf{i} + x^2y\,\mathbf{j}$, the line integral $\int_C \mathbf{F}\cdot d\mathbf{R}$ along any path C joining (0, 0) to (2, 4) has the same value. This is not true for every $\mathbf{F}$ (see Problem 58) but when it is true, the line integral is said to be **independent of path** or **path independent**. Path independence is an important feature of certain vector fields and will be discussed in detail in the next section.

Note: The "line" integral should be thought of as the "curve" integral. The equation(s) of a curve are needed so that the integral can be expressed in terms of a single variable.

APPLICATIONS OF LINE INTEGRALS: MASS AND WORK

Line integrals were developed in the nineteenth century, primarily to deal with problems in physics involving force, fluid flow, electricity, and magnetism. We will show how line integration can be used to compute the mass of a thin wire and the work performed by an object moving along a curve in a force field. Additional applications are examined in the problem set and in Sections 13.3 and 13.4.

Consider a thin wire with the shape of a curve C and let $\delta(x, y, z)$ be the density (mass per unit length) at each point $P(x, y, z)$ on the wire. Suppose the curve is described by parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \leq t \leq b$, and subdivide the parameter interval $[a, b]$ into n equal parts. This induces a partition of the portion of the curve covered by the wire into n subarcs. Let Δs_k be the length of the kth subarc, and let $P_k(x_k^*, y_k^*, z_k^*)$ be a point chosen arbitrarily from this subarc. Then, the mass of the subarc is approximately

$$\Delta m_k = \delta(x_k^*, y_k^*, z_k^*)\Delta s_k$$

and the sum $\sum_{k=1}^{n} \Delta m_k$ approximates the total mass of the wire. We improve the approximation by taking more and more subdivision points, and the actual mass of the wire is given by the limiting value

$$m = \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k^*, y_k^*, z_k^*)\Delta s_k = \int_C \delta(x, y, z)\, ds$$

The **center of mass** of the wire is then the point $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{1}{m}\int_C x\delta(x, y, z)\, ds \qquad \overline{y} = \frac{1}{m}\int_C y\delta(x, y, z)\, ds \qquad \overline{z} = \frac{1}{m}\int_C z\delta(x, y, z)\, ds$$

EXAMPLE 7 Computing the mass of a thin wire using line integration

A wire has the shape of the curve

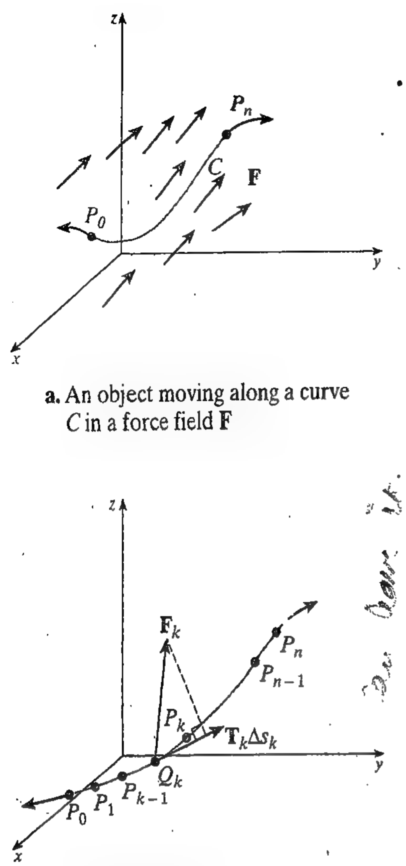
$$x = \sqrt{2}\sin t \qquad y = \cos t \qquad z = \cos t \qquad \text{for } 0 \leq t \leq \pi$$

If the wire has density $\delta(x, y, z) = xyz$ at each point (x, y, z), what is its mass?

Solution

We note that $x'(t) = \sqrt{2}\cos t, y'(t) = z'(t) = -\sin t$. The mass is given by the line integral

$$\begin{aligned}
m &= \int_C \delta(x, y, z)\, dx \\
&= \int_C xyz\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt \\
&= \int_C \sqrt{2}\sin t\cos^2 t\sqrt{(\sqrt{2}\cos t)^2 + (-\sin t)^2 + (-\sin t)^2}\, dt \\
&= \int_0^\pi \sqrt{2}\sin t\cos^2 t\sqrt{2(\cos^2 t + \sin^2 t)}\, dt \\
&= 2\int_0^\pi \cos^2 t\sin t\, dt \\
&= \frac{4}{3}
\end{aligned}$$



a. An object moving along a curve C in a force field **F**

b. The work performed as the object moves along the kth subarc is $W_k = \mathbf{F}_k \cdot \mathbf{T}_k \Delta s_k$

Figure 13.12 Work performed as an object moves in a force field **F** along a curve C

In Problem 49, you are asked to determine the center of mass of this wire. Note that the center of mass may not lie on the curve, or wire, itself. ■

One of the most important physical applications of line integration is in computing work. Recall from Section 9.2, that if an object moves along a line with displacement **D** in a constant force field **F**, the work done is $W = \mathbf{F} \cdot \mathbf{D}$. We now consider the case where **F** is a variable force field and the object moves along an orientable curve C. Assume that C is parameterized by $\mathbf{R}(t)$ and that the object moves in the direction of increasing t. Partition C with subdivision points $P_0, P_1, \dots, P_n$, as shown in Figure 13.12.

For $k = 1, 2, \dots, n$, let $Q_k(x_k^*, y_k^*, z_k^*)$ be a point chosen arbitrarily from the kth subarc C_k (the one with endpoints P_{k-1} and P_k), and let $\mathbf{F}_k = \mathbf{F}(x_k^*, y_k^*, z_k^*)$. If the length Δs_k of the subarc C_k is small, the force will be approximately constant and we assume it has the constant value $\mathbf{F}_k$ on the subarc. Also, the direction of motion will not change much over the subarc, so we can assume that the object will move a distance of Δs_k in the direction of the unit tangent $\mathbf{T}_k = \mathbf{T}(x_k^*, y_k^*, z_k^*)$, for a linear displacement of $\mathbf{T}_k \Delta s_k$. Therefore, we can approximate the work performed over the kth subarc by

$$W_k \approx \mathbf{F}_k \cdot \mathbf{T}_k \Delta s_k$$

By adding the contributions along all n subarcs, we obtain the sum $\sum_{k=1}^{n} \mathbf{F}_k \cdot \mathbf{T}_k \Delta s_k$ as an approximation to the total work performed as the object moves along C in the force field **F**. As the length of the largest subarc $\|\Delta s\|$ tends to 0, this approximating sum approaches the value of the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$; that is,

$$W = \lim_{\|\Delta s\| \to 0} \sum_{k=1}^{n} \mathbf{F}_k \cdot \mathbf{T}_k \Delta s_k = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

These observations lead us to consider **work as a line integral.**

Work as a Line Integral

Let **F** be a continuous force field over a domain D. Then the work W performed as an object moves along a smooth curve C in D is given by the integral

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

where **T** is the unit tangent at each point on C.

Recall that $\mathbf{T} = \dfrac{d\mathbf{R}}{ds}$ where **R** is the position vector of the object moving on C. Thus, the work can also be given by the line integral

$$W = \int_C \mathbf{F} \cdot \frac{d\mathbf{R}}{ds}\, ds = \int_C \mathbf{F} \cdot d\mathbf{R}$$

This form of the line integral for work is used in the following example.

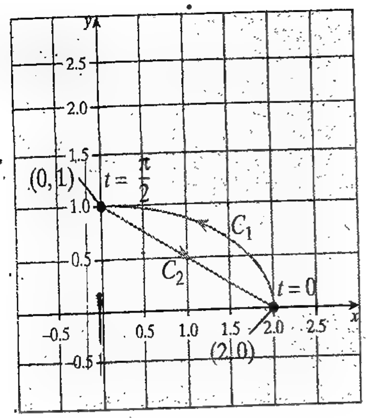


Figure 13.13 The curve C

EXAMPLE 8 Work as a line integral

An object moves in the force field $\mathbf{F} = y^2\mathbf{i} + 2(x+1)y\mathbf{j}$. How much work is performed as the object moves from the point $(2, 0)$ counterclockwise along the elliptical path $x^2 + 4y^2 = 4$ to $(0, 1)$, then back to $(2, 0)$ along the line segment joining the two points, as shown in Figure 13.13.

Solution

If C is the trajectory of the moving object, the work performed is $W = \oint_C \mathbf{F} \cdot d\mathbf{R}$. Let C_1 be the top (elliptical) part of C, and let C_2 be the bottom (linear) part.

The curve C_1 can be parameterized by the equations

$$x = 2\cos t \qquad y = \sin t \qquad \text{for } 0 \le t \le \frac{\pi}{2}$$

(since $x^2 + 4y^2 = 4$), so the position vector for C_1 is $\mathbf{R}_1 = \langle 2\cos t, \sin t\rangle$. Thus, we have $\mathbf{R}_1' = \langle -2\sin t, \cos t\rangle$ and the work performed as the object moves along C_1 is

$$\begin{aligned}
W_1 = \int_{C_1} \mathbf{F} \cdot d\mathbf{R}_1 &= \int_0^{\pi/2} \mathbf{F}[\mathbf{R}_1(t)] \cdot \mathbf{R}_1'(t)\,dt \\
&= \int_0^{\pi/2} [(\sin^2 t)(-2\sin t) + 2(2\cos t + 1)\sin t(\cos t)]\,dt \\
&= \int_0^{\pi/2} (-2\sin^3 t + 4\cos^2 t\sin t + 2\sin t\cos t)\,dt \\
&= \int_0^{\pi/2} (-2\sin^2 t + 4\cos^2 t + 2\cos t)\sin t\,dt \\
&= \int_0^{\pi/2} (6\cos^2 t + 2\cos t - 2)\sin t\,dt \\
&= -\int_1^0 (6u^2 + 2u - 2)\,du \\
&= 1
\end{aligned}$$

Let $u = \cos t$; $du = -\sin t\,dt$. If $t = 0$, $u = 1$, and if $t = \frac{\pi}{2}$, $u = 0$.

For the line segment C_2, a parameterization is

$$x = 2t \qquad y = 1 - t \qquad 0 \le t \le 1$$

The position vector is $\mathbf{R}_2 = \langle 2t, 1 - t\rangle$, so $\mathbf{R}_2' = \langle 2, -1\rangle$ and the work performed as the object moves along C_2 is

$$\begin{aligned}
W_2 = \int_{C_2} \mathbf{F} \cdot d\mathbf{R}_2 &= \int_0^1 \mathbf{F}[\mathbf{R}_2(t)] \cdot \mathbf{R}_2'(t)\,dt \\
&= \int_0^1 [(1-t)^2(2) + 2(2t+1)(1-t)(-1)]\,dt \\
&= \int_0^1 [6t^2 - 6t]\,dt \\
&= -1
\end{aligned}$$

Thus, the total work performed as the object moves along the curve $C = C_1 \cup C_2$ is

$$W = \oint_C \mathbf{F} \cdot d\mathbf{R} = \int_{C_1} \mathbf{F} \cdot d\mathbf{R} + \int_{C_2} \mathbf{F} \cdot d\mathbf{R} = 1 + (-1) = 0 \qquad \blacksquare$$

It can be shown that the work performed as an object moves around any closed path in the force in Example 8 will always be 0. When this occurs, the force is said to be **conservative**. We will discuss conservative force fields in Sections 13.3 and 13.4.

13.2 PROBLEM SET

§ 1–48

A 1. **WHAT DOES THIS SAY?** Explain the difference between $\int_C f\,ds$ and $\int_C f\,dx$.

2. **WHAT DOES THIS SAY?** Discuss the evaluation of the line integral $\int_C \mathbf{F}\cdot d\mathbf{R}$. $= \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$

In Problems 3–12, evaluate the given line integral over the curve C with the prescribed parameterization.

3. $\int_C \frac{1}{3+y}\,ds$ for C: $x = 2t^{3/2}$, $y = 3t$, $0 \le t \le 1$

4. $\int_C (3x - 2y)\,ds$ for C: $x = \sin t$, $y = \cos t$, $0 \le t \le \pi$ 6

5. $\int_C \frac{y^2}{x^3}\,ds$ for C: $x = 2t$, $y = t^4$, $0 \le t \le 1$

6. $\int_C (x^2 + y^2)\,ds$ for C: $x = e^{-t}\cos t$, $y = e^{-t}\sin t$, $0 \le t \le \frac{\pi}{2}$

7. $\int_C (-y\,dx + x\,dy)$ for C: $y = 4x^2$ from $(-1, 4)$ to $(0, 0)$ $\frac{4}{3}$

8. $\int_C (-y\,dx + 3x\,dy)$ for C: $y^2 = x$ from $(1, 1)$ to $(9, 3)$ $\frac{26}{3}$

9. $\int_C (x\,dy - y\,dx)$ for C: $2x - 4y = 1$, $4 \le x \le 8$

10. Evaluate $\int_C [(y - x)\,dx + x^2y\,dy]$, where C is the curve defined by $y^2 = x^3$ from $(1, -1)$ to $(1, 1)$. 4/15

11. Evaluate $\int_C [(x + y)^2\,dx - (x - y)^2\,dy]$, where C is the curve defined by $y = |2x|$ from $(-1, 2)$ to $(1, 2)$. $\frac{26}{3}$

12. Evaluate $\int_C [(y^2 - x^2)\,dx - x\,dy]$, where C is the quarter-circle $x^2 + y^2 = 4$ from $(0, 2)$ to $(2, 0)$. $\frac{8}{3}+\pi$

13. Evaluate $\int_C [(x^2 + y^2)\,dx + 2xy\,dy]$ for these choices of the curve C:
   a. C is the quarter circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$. $-\frac{1}{3}$
   b. C is the straight line $y = 1 - x$ from $(1, 0)$ to $(0, 1)$. $-\frac{1}{3}$

14. Evaluate $\int_C [x^2y\,dx + (x^2 - y^2)\,dy]$ for these choices of the curve C:
   a. C is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$.
   b. C is the segment of the line $y = 2x$ for $0 \le x \le 2$.

15. Evaluate the integral in Problem 14 for the path C that consists of the horizontal line segment $(0, 0)$ to $(2, 0)$, followed by the vertical segment from $(2, 0)$ to $(2, 4)$.

16. Evaluate $\int_C (-xy^2\,dx + x^2\,dy)$, where C is the path shown in Figure 13.14.

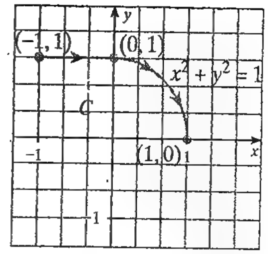


Figure 13.14 Path of C for Problem 16

17. Evaluate $\int_C (-y^2\,dx + x^2\,dy)$, where C is the path shown in Figure 13.15.

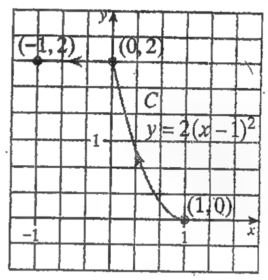


Figure 13.15 Path of C for Problem 17

18. Evaluate $\oint_C [(x^2 - y^2)\,dx + x\,dy]$, where C is the circular path given by $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$. 4π

19. Evaluate $\oint_C (x^2y\,dx - xy\,dy)$, where C is the path that begins at $(0, 0)$, goes to $(1, 1)$ along the parabola $y = x^2$, and then returns to $(0, 0)$ along the line $y = x$.

In Problems 20–22, evaluate $\int_C \mathbf{F}\cdot d\mathbf{R}$, *where* $\mathbf{F} = (5x + y)\mathbf{i} + x\mathbf{j}$ *and C is the specified curve.*

20. C is the straight line segment from $(0, 0)$ to $(2, 1)$. 12

21. C is the curve given by $\mathbf{R}(t) = 2t\mathbf{i} + t\mathbf{j}$ for $0 \le t \le 1$. 12

22. C is the vertical line from $(0, 0)$ to $(0, 1)$, followed by the horizontal line from $(0, 1)$ to $(2, 1)$. 12

Evaluate the line integrals in Problems 23–38.

23. $\int_C \mathbf{F}\cdot d\mathbf{R}$, where $\mathbf{F} = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}} \right\rangle$ and C is the quarter circle path $x^2 + y^2 = a^2$, traversed from $(a, 0)$ to $(0, a)$ $-a$

24. $\int_C (y\,dx - x\,dy + dz)$, where C is the helical path given by
   a. $x = 3\sin t$, $y = 3\cos t$, $z = t$ for $0 \le t \le \frac{\pi}{2}$ 5π
   b. $x = a\sin t$, $y = a\cos t$, $z = t$ for constant a and $0 \le t \le \frac{\pi}{2}$

25. $\int_C (x\,dx + y\,dy + z\,dz)$, where C is the following path:
   a. the helix defined by $x = \cos t$, $y = \sin t$, $z = t$ for $0 \le t \le \frac{\pi}{2}$ $\frac{\pi^2}{8}$
   b. the straight line segment from $(1, 0, 0)$ to $(0, 1, \frac{\pi}{2})$ $\frac{\pi^2}{8}$

26. $\int_C (-y\,dx + x\,dy + xz\,dz)$, where C is the following path:

a. the helix defined by $x = \cos t$, $y = \sin t$, $z = t$ for $0 \le t \le 2\pi$ 2π

b. the unit circle $x^2 + y^2 = 1$, $z = 0$, traversed once counterclockwise as viewed from above 2π

27. $\int_C (5xy\,dx + 10yz\,dy + z\,dz)$, where C is the following path:

a. the parabolic arc $x = y^2$ from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the line segment given by $x = 1$, $y = 1$, $0 \le z \le 1$ $5/2$

b. the straight line segment from $(0, 0, 0)$ to $(1, 1, 1)$ $11/2$

28. $\int_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = (y - 2z)\mathbf{i} + x\mathbf{j} - 2xy\mathbf{k}$ and C is the path given by $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} - \mathbf{k}$ for $1 \le t \le 2$ 9

29. $\oint_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = x\mathbf{i} + xy\mathbf{j} + x^2yz\mathbf{k}$, and C is the elliptical path given by $x^2 + 4y^2 - 8y + 3 = 0$ in the xy-plane, traversed once counterclockwise as viewed from above $\pi/2$

30. $\oint_C [(y + z)\,dx + (x + z)\,dy + (x + y)\,dz]$, where C is the circle of radius 1 centered on the z-axis in the plane $z = 2$, traversed once counterclockwise as viewed from above 0

31. $\int_C 2xy^2z\,ds$ for C: $x = t$, $y = t^2$, $z = \frac{2}{3}t^3$ for $0 \le t \le 1$ $11/6$ $29/7$

32. $\int_C ye^{xz}\,ds$, where C is the line segment from $(0, 0, 0)$ to $(2, 1, 3)$ $\frac{\sqrt{14}}{12}(e^6-1)$

33. $\oint_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} - (x + z)\mathbf{k}$ and C is the boundary of the triangle with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, traversed once clockwise, as viewed from above $-\frac{1}{3}$

34. $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$, where $\mathbf{F} = -3y\mathbf{i} + 3x\mathbf{j} + 3x\mathbf{k}$ and C is the straight line segment from $(0, 0, 1)$ to $(1, 1, 1)$ 0

35. $\oint_C \mathbf{F} \cdot \mathbf{T}\,ds$, where $\mathbf{F} = -x\mathbf{i} + 2\mathbf{j}$, and C is the boundary of the trapezoid with vertices $(0, 0)$, $(1, 0)$, $(2, 1)$, $(0, 1)$, traversed once clockwise as viewed from above 0

36. $\int_C y\,ds$, where C is the curve given by $\mathbf{R}(t) = t\mathbf{i} + 2t^3\mathbf{j}$, $0 \le t \le 2$ $\frac{1}{108}(577^{3/2}-1)$

37. $\int_C (x + y)\,ds$, where C is given by $\mathbf{R}(t) = (\cos^2 t)\mathbf{i} + (\sin^2 t)\mathbf{j}$, $-\frac{\pi}{4} \le t \le 0$ $\frac{\sqrt{2}}{2}$

38. $\int_C \frac{x^2 + xy + y^2}{z^2}\,ds$, where C is the path given by $\mathbf{R}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k}$ for $0 \le t \le 2\pi$ 2π

B 39. Evaluate the line integral

$$\oint_C \frac{x\,dy - y\,dx}{x^2 + y^2}$$ 2π

where C is the unit circle $x^2 + y^2 = 1$ traversed once counterclockwise.

40. $\oint_C \frac{dx + dy}{|x| + |y|}$, where C is the square $|x| + |y| = 1$, traversed once counterclockwise. 0

41. How much work is done by a constant force $\mathbf{F} = a\mathbf{i} + \mathbf{j}$ when a particle moves along the line $y = ax$ from $x = a$ to $x = 0$? $-2a^2$

42. A force field in the plane is given by $\mathbf{F} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$. Find the total work done by this force in moving a point mass counterclockwise around the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$. 16

43. Find the work done by the force field $\mathbf{F} = (x^2 + y^2)\mathbf{i} + (x + y)\mathbf{j}$ as an object moves counterclockwise along the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$, and then back to $(1, 0)$ along the x-axis. $\frac{\pi}{2} - \frac{4}{3}$

44. A force acting on a point mass located at (x, y) is given by $\mathbf{F} = y\mathbf{i} + 2x\mathbf{j}$. Find the work done by this force as the point mass moves along a straight line from $(1, 0)$ to $(0, 1)$. $1/2$

Find the work done by the force $\mathbf{F}(x, y, z)$ on an object moving along the curve C in Problems 45–48.

45. $\mathbf{F} = (y^2 - z^2)\mathbf{i} + 2yz\mathbf{j} - x^2\mathbf{k}$, and C is the path given by $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$, for $0 \le t \le 1$. $1/35$

46. $\mathbf{F} = 2xy\mathbf{i} + (x^2 + 2)\mathbf{j} + y\mathbf{k}$, and C is the line segment from $(1, 0, 2)$ to $(3, 4, 1)$. 42

47. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (xz - y)\mathbf{k}$, and C is the line segment from $(0, 0, 0)$ to $(2, 1, 2)$. $25/6$

48. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (xz - y)\mathbf{k}$, and C is the path given by $\mathbf{R}(t) = t^2\mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k}$ for $0 \le t \le 1$. $5/2$

49. A wire has the shape of the curve C: $x = \sqrt{2}$, $y = \cos t$, $z = \cos t$ for $0 \le t \le \pi$, and the density at the point (x, y, z) on the curve is $\delta(x, y, z) = xyz$. The mass of this wire was computed in Example 7. Find the center of mass.

50. Find the center of mass of a wire in the shape of the helix $x = 3\sin t$, $y = 3\cos t$, $z = 2t$ for $0 \le t \le \pi$ and the following choices of density $\delta(x, y, z)$:

a. $\delta(x, y, z) = z$ b. $\delta(x, y, z) = x$

51. Find the centroid of a thin wire in the shape of the curve $x = 2t$, $y = t^2$, $\int_C \mathbf{E} \cdot d\mathbf{R}$, for $0 \le t \le 2$.

52. Find the centroid of the arch of the cycloid C: $x = t - \sin t$, $y = 1 - \cos t$ for $0 \le t \le 2\pi$. *Note*: The centroid may be thought of as the center of mass of a wire of constant density.

53. A 180-lb laborer carries a bag of sand weighing 40 lb up a circular helical staircase (Figure 13.16) on the outside of a tower 50 ft high and 20 ft in diameter. How much work is done as the laborer climbs to the top in exactly five revolutions?

Figure 13.16 A circular helical staircase

54. Repeat Problem 53, assuming that the bag leaks 1 lb of sand for every 10 ft of ascent. How much work is done during the laborer's climb to the top?

55. A 5,000-lb satellite orbits the earth in a circular orbit 5,000 mi from the center of the earth. How much work is done as the satellite moves through one complete revolution?

C 56. Suppose a particle with charge Q and mass m moves with velocity $\mathbf{V}$ under the influence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$. Then the total force on the particle is $\mathbf{F} = Q(\mathbf{E} + \mathbf{V} \times \mathbf{B})$, called the *Lorentz force*. Use Newton's second law of motion, $\mathbf{F} = m\mathbf{A}$, to show that

$$m\frac{d\mathbf{V}}{dt} \cdot \mathbf{V} = Q\mathbf{E} \cdot \frac{d\mathbf{R}}{dt}$$

and then evaluate the line integral $\int_C \mathbf{E} \cdot d\mathbf{R}$, where C is the trajectory of a particle traveling with constant speed.

57. **Counterexample Problem** If $\int_C f(x, y, z)\, ds = 0$, is it true that $f(x, y, z) = 0$ on C? Either prove that it is, or find a counterexample.

58. **Counterexample Problem** This problem shows that not all line integrals are path independent. Let $\mathbf{F} = \langle y, -x \rangle$, and let C_1 and C_2 be the following two paths joining (0, 0) to (1, 1).

$$C_1\colon y = x \text{ for } 0 \le x \le 1 \quad \text{and} \quad C_2\colon y = x^2 \text{ for } 0 \le x \le 1$$

Show that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{R} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{R}$$

13.3 The Fundamental Theorem and Path Independence

IN THIS SECTION fundamental theorem for line integrals, conservative vector fields, independence of path

In general, the value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ depends on the path of integration C, but in certain cases, the integral will be the same for all paths in a given region D with the same initial point P and terminal point Q. In this case, we say the line integral is **independent of path in D** (see Figure 13.17). In this section, we will study path independence and characterize the kind of vector field $\mathbf{F}$ for which it occurs.

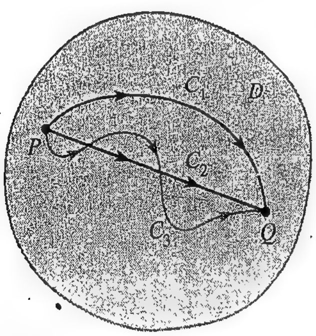


Figure 13.17 A line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path in D if its value is the same for all curves joining any two points in D.

FUNDAMENTAL THEOREM FOR LINE INTEGRALS

The fundamental theorem of calculus (Section 5.4) says that if the function f is continuous on $[a, b]$, then

$$\boxed{\int_a^b f(x)\, dx = F(b) - F(a)}$$

where F is any antiderivative of f; that is, $F'(x) = f(x)$. For a function of two or three variables, the analogue of the derivative is the gradient, and the corresponding analogue of the fundamental theorem is the following theorem.

***THEOREM 13.2* Fundamental theorem for line integrals**

No proof

Let C be a piecewise smooth curve that is parameterized by the vector function $\mathbf{R}(t)$ for $a \le t \le b$, and let $\mathbf{F}$ be a vector field that is continuous on C. If f is a scalar function such that $\boxed{\mathbf{F} = \nabla f}$, then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{R} = f(Q) - f(P)}$$

where $Q = \mathbf{R}(b)$ and $P = \mathbf{R}(a)$ are the endpoints of C.

Proof We will prove this theorem for the case where $f(x, y, z)$ is a function of three variables, such that $\mathbf{F} = \nabla f(x, y, z)$. Suppose $\mathbf{R}(t) = \langle x(t), y(t), z(t) \rangle$ and let G be the composite function $G(t) = f[x(t), y(t), z(t)]$. Then, according to the chain rule

$$\frac{dG}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

and we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{R} &= \int_C \nabla f \cdot d\mathbf{R} \\
&= \int_C \left[\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz\right] \\
&= \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right] dt \\
&= \int_a^b \frac{dG}{dt}\,dt && \text{Substitution} \\
&= G(b) - G(a) && \text{Fundamental theorem of calculus} \\
&= f[x(b), y(b), z(b)] - f[x(a), y(a), z(a)] && \text{Substitution} \\
&= f[R(b)] - f[R(a)] && \text{Substitution} \\
&= f(Q) - f(P)
\end{aligned}$$

□

EXAMPLE 1 Using the fundamental theorem to evaluate a line integral

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$, where

$$\mathbf{F} = \nabla(e^x \sin y - xy - 2y)$$

and C is the path described by $\mathbf{R}(t) = \left[t^3 \sin \frac{\pi}{2}t\right]\mathbf{i} - \left[\frac{\pi}{2}\cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)\right]\mathbf{j}$ for $0 \le t \le 1$.

Solution

First, note that the hypotheses of the fundamental theorem for line integrals are satisfied since

$$f(x, y) = e^x \sin y - xy - 2y$$

has continuous partial derivatives on the smooth curve C. At the endpoints of C, we find

left endpoint $(t = 0)$; $\mathbf{R}(0) = \langle 0, 0 \rangle$

$$f(0, 0) = e^0 \sin 0 - 0 - 0 = 0$$

right endpoint $(t = 1)$; $\mathbf{R}(1) = \left\langle 1, \frac{\pi}{2} \right\rangle$

$$f\left(1, \tfrac{\pi}{2}\right) = e^1 \sin \tfrac{\pi}{2} - \tfrac{\pi}{2} - 2\left(\tfrac{\pi}{2}\right) = e - \tfrac{3\pi}{2}$$

Thus, according to the fundamental theorem for line integrals, we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{R} &= f(Q) - f(P) = f\left(1, \tfrac{\pi}{2}\right) - f(0, 0) \\
&= \left(e - \tfrac{3\pi}{2}\right) - 0 = e - \tfrac{3\pi}{2}
\end{aligned}$$

■

CONSERVATIVE VECTOR FIELDS

A key requirement for evaluating the line integral $\int_C \mathbf{F}\cdot d\mathbf{R}$ by the fundamental theorem of line integrals is that $\mathbf{F}$ be the gradient of some scalar function f, that is, $\mathbf{F} = \nabla f$. Our next goal is to learn how to determine when a given vector field $\mathbf{F}$ can be expressed as a gradient field, and we begin by introducing some terminology.

Conservative Vector Field

Def'n

A vector field $\mathbf{F}$ is said to be **conservative** in a region D if $\mathbf{F} = \nabla f$ for some scalar function f in D. The function f is called a **scalar potential** of $\mathbf{F}$ in D. That is,

$$\underbrace{\mathbf{F}}_{\text{Conservative vector field}} = \nabla \underbrace{f}_{\text{Scalar potential}} \quad \text{for } (x, y) \text{ on } D$$

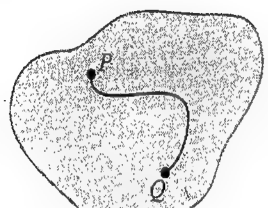
The term *conservative* comes from physics and is related to the law of conservation of energy (see Problem 53).

EXAMPLE 2 Verifying that a vector field is conservative

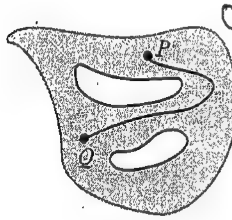
Verify that the vector field $\mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j}$ is conservative, with scalar potential $f = x^2y$.

Solution

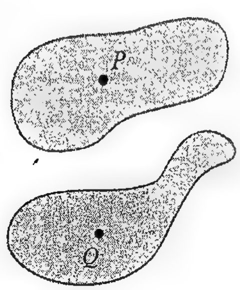
$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$ and this is the same as $\mathbf{F}$, so $\mathbf{F}$ is conservative. ■



a. A region that is simply connected (no holes)



b. A connected region that is *not* simply connected



c. A region that is not connected

Figure 13.18 Regions that are connected and not connected

Conservative criteria

It is one thing to verify that a vector field is conservative as we did in Example 2, but it is more common to be given a vector field $\mathbf{F}$ and then be asked to determine whether it is conservative without knowing the answer in advance. We will establish a simple criterion for $\mathbf{F}$ to be conservative on any set D in the plane that has these properties:

1. Any two points P and Q in D can be joined by a piecewise-smooth curve entirely within D.
2. Every closed curve in D encloses only points that are also in D.

A region D with property (1) is **connected**, and a connected region with property (2) is **simply connected**. Roughly speaking, a simply connected region is one with no "holes," as shown in Figure 13.18.

Criteria

THEOREM 13.3 Cross-partials test for a conservative vector field in the plane

Consider the vector field $\mathbf{F}(x, y) = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$, where u and v have continuous first partials in the open, simply connected region D in the plane. Then $\mathbf{F}(x, y)$ is conservative in D if and only if

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad \text{throughout } D$$

Proof We outline the proof in Problem Set 13.4 after our discussion of Green's theorem. ❑

EXAMPLE 3 Finding a scalar potential function

Show that the vector field $\mathbf{F} = (e^x \sin y - y)\mathbf{i} + (e^x \cos y - x - 2)\mathbf{j}$ is conservative and then find a scalar potential function f for $\mathbf{F}$.

Solution

First note that the component functions $u(x, y) = e^x \sin y - y$ and $v(x, y) = e^x \cos y - x - 2$ have continuous partial derivatives. Then

$$\frac{\partial u}{\partial y} = e^x \cos y - 1 \quad \text{and} \quad \frac{\partial v}{\partial x} = e^x \cos y - 1$$

Since $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$, it follows that **F** is conservative. To find a scalar potential function f such that $\nabla f = \mathbf{F}$, we note that f must satisfy $u(x, y) = f_x(x, y)$ and $v(x, y) = f_y(x, y)$.

$$f(x, y) = \int u(x, y)\, dx = \underbrace{\int (e^x \sin y - y)\, dx}$$

This is the "partial integral" in the sense that y is held constant while the integration is performed with respect to x alone.

$$= e^x \sin y - xy + k(y)$$

Note that $k(y)$ is a function of y alone—a "constant" as far as x-integration is concerned.

Because f must also satisfy $f_y(x, y) = v(x, y)$, we compute the partial derivative of this f with respect to y:

$$f_y(x, y) = \frac{\partial}{\partial y}[e^x \sin y - xy + k(y)] = e^x \cos y - x + \frac{dk}{dy}$$

Set this equal to $v = e^x \cos y - x - 2$ and solve for $\dfrac{dk}{dy}$:

$$\begin{aligned} e^x \cos y - x + \frac{dk}{dy} &= e^x \cos y - x - 2 \\ \frac{dk}{dy} &= -2 \\ k(y) &= -2y + C \end{aligned}$$

Thus, any function $f(x, y) = e^x \sin y - xy - 2y + C$. Any such function is a scalar potential of **F** and, for simplicity, we pick $C = 0$:

$$\boxed{f(x, y) = e^x \sin y - xy - 2y}$$ ■

In Example 3, we began by using the fact that $u = f_x$. In general, the issue of whether to start with $u = f_x$ or $v = f_y$ is often determined by which equation leads to the simpler integration.

EXAMPLE 4 Testing for a conservative vector field in the plane

Determine whether the vector field $\mathbf{F} = ye^{xy}\mathbf{i} + (xe^{xy} + x)\mathbf{j}$ is conservative; if it is, find a scalar potential.

Solution

We have $u(x, y) = ye^{xy}$ and $v(x, y) = xe^{xy} + x$.

$$\frac{\partial u}{\partial y} = xye^{xy} + e^{xy} \qquad \frac{\partial v}{\partial x} = xye^{xy} + e^{xy} + 1$$

so $\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$, and **F** is not conservative. ■

In $\mathbb{R}^3$, the following generalization of the cross-partials test may be used as a criterion for determining whether a given vector field is conservative.

THEOREM 13.4 **The curl criterion for a conservative vector field in $\mathbb{R}^3$**

Criteria II

Suppose the vector field **F** and curl **F** are both continuous in the simply connected region D of $\mathbb{R}^3$. Then **F** is conservative in D if and only if curl $\mathbf{F} = \mathbf{0}$.

Proof As in $\mathbb{R}^2$, a simply connected region in $\mathbb{R}^3$ can be described informally as one with no "holes." A proof using Stokes' theorem is given in Section 13.6 ❑

Note that a vector field $\mathbf{F} = \langle u(x, y), v(x, y)\rangle$ in $\mathbb{R}^2$ can be regarded as the vector field $\mathbf{G} = \langle u(x, y, 0), v(x, y, 0), 0\rangle$ in $\mathbb{R}^3$. Since

$$\begin{aligned}\operatorname{curl}\mathbf{G} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u(x, y, 0) & v(x, y, 0) & 0 \end{vmatrix} \\ &= 0\mathbf{i} + 0\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}\end{aligned}$$

we have curl $\mathbf{G} = \mathbf{0}$ if and only if $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$. Thus, Theorem 13.4 becomes the cross-partials test if **F** is in $\mathbb{R}^2$.

EXAMPLE 5 Determining a scalar potential for a conservative vector field in $\mathbb{R}^3$

Show that the vector field

$$\mathbf{F} = \langle 20x^3z + 2y^2, 4xy, 5x^4 + 3z^2\rangle$$

is conservative in $\mathbb{R}^3$ and find a scalar potential function for **F**.

Solution

To show that **F** is conservative in $\mathbb{R}^3$, note that the components of **F** are continuous with continuous partial derivatives and that

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 20x^3z + 2y^2 & 4xy & 5x^4 + 3z^2 \end{vmatrix} \\ &= (0 - 0)\mathbf{i} - (20x^3 - 20x^3)\mathbf{j} + (4y - 4y)\mathbf{k} \\ &= \mathbf{0}\end{aligned}$$

Therefore, according to Theorem 13.4, the vector field **F** is conservative.

To determine a scalar potential function for **F**, we proceed as in Example 3, except that now we have three component equations, each involving three variables:

$$\frac{\partial f}{\partial x} = 20x^3z + 2y^2 \qquad \frac{\partial f}{\partial y} = 4xy \qquad \frac{\partial f}{\partial z} = 5x^4 + 3z^2$$

Integrating the first equation with respect to x (holding y and z constant), we obtain

$$f(x, y, z) = 5x^4z + 2xy^2 + g(y, z)$$

where $g(y, z)$ is a constant with respect to x-integration. Taking the partial derivative of this expression with respect to y, and then comparing the result to the required equation $\frac{\partial f}{\partial y} = 4xy$, we see that

$$4xy + \frac{\partial g}{\partial y} = 4xy$$

Thus,

$$\frac{\partial g}{\partial y} = 0 \quad \text{and} \quad g(y, z) = h(z)$$

where $h(z)$ is a constant with respect to x- and y-integration, so

$$f(x, y, z) = 5x^4 z + 2xy^2 + h(z).$$

Finally, we compute the partial derivative of f with respect to z from this equation, and compare it to the required equation $\frac{\partial f}{\partial z} = 5x^4 + 3z^2$:

$$\begin{aligned} 5x^4 + h'(z) &= 5x^4 + 3z^2 \\ h'(z) &= 3z^2 \\ h(z) &= z^3 + C \end{aligned}$$

Therefore, any function of the form

$$f(x, y, z) = 5x^4 z + 2xy^2 + z^3 + C$$

is a scalar potential for the vector field $\mathbf{F}$. ■

INDEPENDENCE OF PATH

We now have the tools to discuss path independence for a line integral. Here are the definition and notation we will use.

Independence of Path

> The line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ is **independent of path** in a region D if for any two points P and Q in D the line integral along every piecewise smooth curve in D from P to Q has the same value.

The following theorem provides three equivalent ways of determining whether a given line integral is path independent.

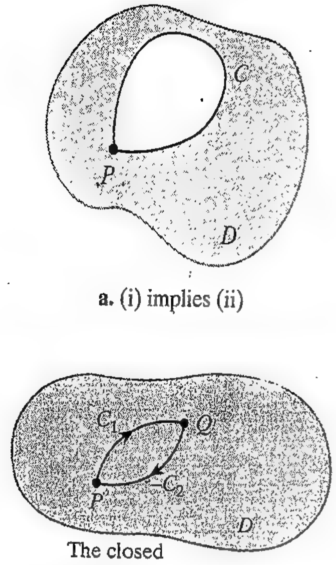
THEOREM 13.5 **Equivalent conditions for path independence**

If $\mathbf{F}$ is a continuous vector field on the open connected set D, then the following three conditions are either all true or all false:

i. $\mathbf{F}$ is conservative on D; that is, $\mathbf{F} = \nabla f$ for some function f defined on D.

ii. $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$ for every piecewise smooth closed curve C in D.

iii. $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path within D.



a. (i) implies (ii)

b. (ii) implies (iii)

Figure 13.19 Equivalent conditions for path independence

note 4 exam

Proof To prove this theorem, it is enough to prove that (i) implies (ii), that (ii) implies (iii), and that (iii) implies (i). Then, if any one is true, all are true.

(i) implies (ii) Assume that $\mathbf{F}$ is conservative, and let C be a closed curve in D, as shown in Figure 13.19a. Then any point P on C can serve as both the initial point and the terminal point of the curve, and according to the fundamental theorem of line integrals, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = f(P) - f(P) = 0$$

where f is a scalar potential for $\mathbf{F}$.

(ii) implies (iii) Let C_1 and C_2 be two curves in D with the same initial point P and terminal point Q. Then the curve C formed by C_1 followed by $-C_2$ (the reverse of C_2) is a closed curve beginning and ending at P. According to condition (ii), the line integral around this closed curve must be 0, so we have

$$\underbrace{0 = \oint_C \mathbf{F} \cdot d\mathbf{R}}_{\text{Condition (ii)}} = \underbrace{\int_{C_1} \mathbf{F} \cdot d\mathbf{R} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{R}}_{\text{Additivity of line integrals}}$$

and it follows that (see Figure 13.19b)

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{R} = -\int_{-C_2} \mathbf{F} \cdot d\mathbf{R} = \int_{C_2} \mathbf{F} \cdot d\mathbf{R}$$

Since C_1 and C_2 were chosen as any two curves in D with the same endpoints, it follows that $\oint_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path in D.

(iii) implies (i) For this implication, we assume that $\oint_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path in D and construct a scalar function f such that $\mathbf{F} = \nabla f$ in order to show that $\mathbf{F}$ must be conservative. Details of this implication are outlined in Problem 54. ❑

EXAMPLE 6 Work along a closed path in a conservative force field

Show that no work is performed when an object moves along a closed path in a connected domain where the force field is conservative.

Solution

In such a force field $\mathbf{F}$, we have $\nabla f = \mathbf{F}$, where f is a scalar potential of $\mathbf{F}$, and because the path of motion is closed, it begins and ends at the same point P. Thus, the work is given by

$$W = \oint_C \mathbf{F} \cdot d\mathbf{R} = f(P) - f(P) = 0 \qquad \blacksquare$$

We now have several ways for evaluating a given line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$. We can

1. Parameterize C and use the parameterization to convert the line integral into an "ordinary" integral in t over an interval $a \le t \le b$.
2. Check to see whether $\mathbf{F}$ is conservative. If it is, find a scalar potential function f and then use the fundamental theorem of line integrals.

3. If $\mathbf{F}$ is conservative, find a convenient path C_1 with the same endpoints as C and use the fact that $\int_{C_1} \mathbf{F} \cdot d\mathbf{R} = \int_C \mathbf{F} \cdot d\mathbf{R}$ since the line integral is independent of path.

Here is an example that illustrates these options.

EXAMPLE 7 Strategy for evaluating a line integral

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$, where

$$\mathbf{F} = [(2x - x^2 y)e^{-xy} + \tan^{-1} y]\mathbf{i} + \left[\frac{x}{y^2+1} - x^3 e^{-xy}\right]\mathbf{j}$$

for each of the following curves:

a. C_1: the ellipse $9x^2 + 4y^2 = 36$

b. C_2: the curve with parametric equations $x = t^2 \cos \pi t, y = e^{-t} \sin \pi t, 0 \leq t \leq 1$

Solution

First, we check to see whether $\mathbf{F}$ is conservative using the cross-partials test:

$$\begin{aligned}\frac{\partial}{\partial x}\left[\frac{x}{y^2+1} - x^3 e^{-xy}\right] &= \frac{1}{y^2+1} + (x^3 y - 3x^2)e^{-xy} \\ &= \frac{\partial}{\partial y}\left[(2x - x^2 y)e^{-xy} + \tan^{-1} y\right]\end{aligned}$$

Thus, $\mathbf{F}$ is conservative and we could answer both questions by finding a potential function for $\mathbf{F}$, but it isn't really necessary.

a. Since $\mathbf{F}$ is conservative, and the ellipse C_1 is a closed curve, we must have $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0.$

b. The curve C_2 has initial point $P(0, 0)$, where $t = 0$, and terminal point $Q(-1, 0)$, where $t = 1$. Since $\mathbf{F}$ is conservative, the line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path, so the given line integral has the same value as $\int_{C_3} \mathbf{F} \cdot d\mathbf{R}$, where C_3 is the line segment from P to Q. A parameterization for C_3 is $x = -t$, $y = 0$. Thus, if $\mathbf{R} = \langle -t, 0 \rangle$, we have $\mathbf{R}' = \langle -1, 0 \rangle$ and

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{R} = \int_{C_3} \mathbf{F} \cdot d\mathbf{R} &= \int_0^1 \mathbf{F}[\mathbf{R}(t)] \cdot \mathbf{R}'(t)\, dt \\ &= \int_0^1 \left\{\left[(2(-t) - 0)e^0 + \tan^{-1} 0\right](-1) + \left[\frac{-t}{0+1} - (-t)^3 e^0\right](0)\right\} dt \\ &= \int_0^1 2t\, dt \\ &= 1\end{aligned}$$

■

In the next section, we will develop another criterion for path independence as part of our study of an important result known as Green's theorem.

Fall 20 – Vy

Guidelines (§1–19, 27, 29, 31–35)

13.3 PROBLEM SET

A 1. **Exploration Problem** What is a conservative vector field?

2. **WHAT DOES THIS SAY?** Describe the fundamental theorem of line integrals.

3. **WHAT DOES THIS SAY?** Explain what is meant by independence of path. Describe various equivalent conditions for path independence.

Determine whether or not each vector field in Problems 4–9 is conservative, and if it is, find a scalar potential.

4. $y^2\mathbf{i} + 2xy\mathbf{j}$ conservative $f(x,y) = y^2x$

5. $2xy^3\mathbf{i} + 3y^2x^2\mathbf{j}$ cons. $f(x,y) = x^2y^3$

6. $(xe^{xy}\sin y)\mathbf{i} + (e^{xy}\cos xy + y)\mathbf{j}$ not cons.

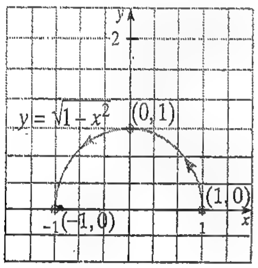
7. $(-y + e^x \sin y)\mathbf{i} + [(x+2)e^x \cos y]\mathbf{j}$ not cons.

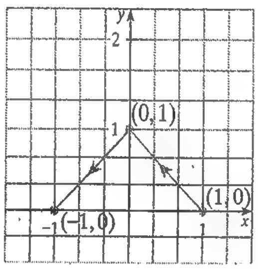
8. $(y - x^2)\mathbf{i} + (2x + y^2)\mathbf{j}$ not cons

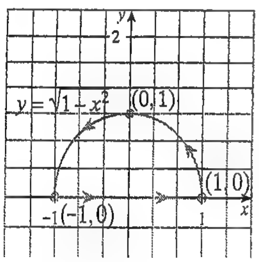
9. $(e^{2x}\sin y)\mathbf{i} + (e^{2x}\cos y)\mathbf{j}$ not cons

Evaluate the line integrals in Problems 10–13 for each of the given paths.

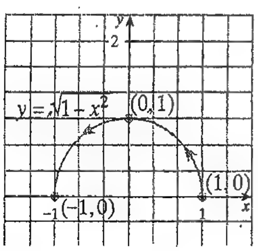
10. $\int_C [(3x + 2y)\,dx + (2x + 3y)\,dy]$ conservative

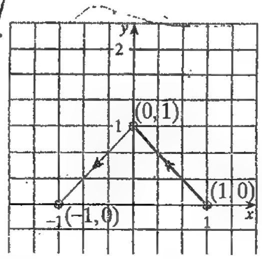
a.  0

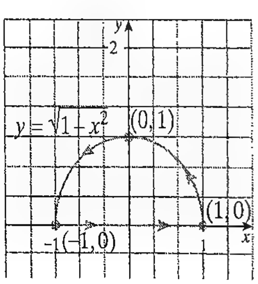
b.  0

c.  0

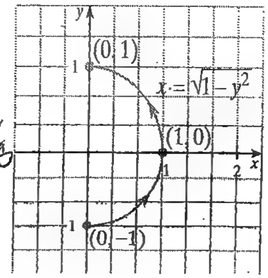
11. $\int_C [(3x + 2y)\,dx - (2x + 3y)\,dy]$

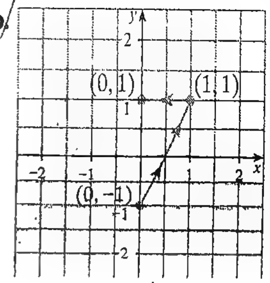
a.  -22

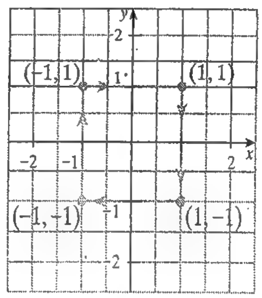
b.  -4

c.  22

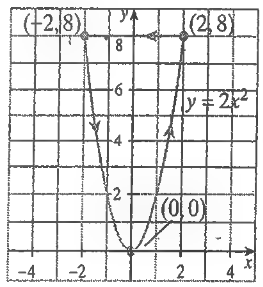
12. $\int_C [2x^2y\,dx + x^3\,dy]$

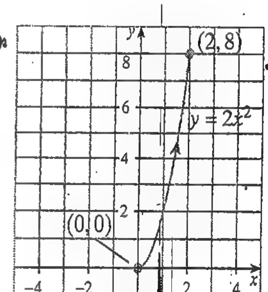
a. 

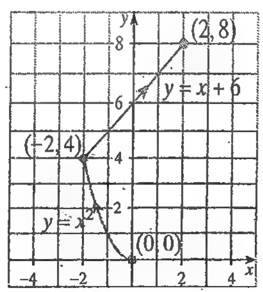
b.  1/6

c.  -4/3

13. $\int_C [2xy\,dx + x^2\,dy]$ conservative

a.  0

b.  32

c.  32

In each of Problems 14–19, show that the vector field **F** *is conservative and find a scalar potential f for* **F**. *Then evaluate the line integral* $\int_C \mathbf{F}\cdot d\mathbf{R}$, *where C is any smooth path connecting* $A(0,0)$ *to* $B(1,1)$.

14. $\mathbf{F}(x,y) = (x + 2y)\mathbf{i} + (2x + y)\mathbf{j}$ $f(x,y) = \frac{x^2}{2} + 2xy + \frac{y^2}{2}$, 3

15. $\mathbf{F}(x,y) = 2xy\mathbf{i} + x^2\mathbf{j}$ $f(x,y) = x^2y$, 1

16. $\mathbf{F}(x,y) = (y - x^2)\mathbf{i} + (x + y^2)\mathbf{j}$ $f(x,y) = yx - \frac{x^3}{3} + \frac{y^3}{3}$, 1

17. $\mathbf{F}(x,y) = (2x - y)\mathbf{i} + (y^2 - x)\mathbf{j}$ $f(x,y) = x^2 - yx + \frac{y^3}{3}$, $\frac{1}{3}$

18. $\mathbf{F}(x,y) = e^{-y}\mathbf{i} - xe^{-y}\mathbf{j}$ $f(x,y) = xe^{-y}$, e^{-1}

19. F(

Show t and fir

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21. e^x

22. yz

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19. $\mathbf{F}(x, y) = \dfrac{(y+1)\mathbf{i} - x\mathbf{j}}{(y+1)^2}$ $g(x,y) = \frac{x}{(y+1)}$? $\frac{1}{2}$

Show that the given vector field $\mathbf{F}$ *in Problems 20–25 is conservative and find a scalar potential function* f *for* $\mathbf{F}$.

20. $yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}$
21. $e^{xy}\,yz\,\mathbf{i} + e^{xy}\,xz\,\mathbf{j} + e^{xy}\,\mathbf{k}$
22. $yz^{-1}\,\mathbf{i} + xz^{-1}\,\mathbf{j} - xyz^{-2}\,\mathbf{k}$
23. $(x^2 + y^2 + z^2)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$
24. $(y \sin z)\mathbf{i} + (x \sin z + 2y)\mathbf{j} + (xy \cos z)\mathbf{k}$
25. $(xy^2 + yz)\mathbf{i} + (x^2y + xz + 3y^2z)\mathbf{j} + (xy + y^3)\mathbf{k}$

In Problems 26–29, show that the vector field $\mathbf{F}$ *is conservative and evaluate* $\int_C \mathbf{F} \cdot d\mathbf{R}$ *for any piecewise smooth path joining* $A(1, 0, -1)$ *to* $B(0, -1, 1)$.

26. $\mathbf{F}(x, y, z) = \langle 3x^2y^2z, 2x^3yz, x^3y^2 - e^{-z} \rangle$
27. $\mathbf{F}(x, y, z) = \langle \sin z, -z \sin y, x \cos z + \cos y \rangle$ cos 1 + tan 1 + 1
28. $\mathbf{F}(x, y, z) = \langle 2xz^3 - e^{-xy} y \sin z, -xe^{-xy} \sin z, 3x^2z^2 + e^{-xy} \cos z \rangle$
29. $\mathbf{F}(x, y, z) = \left\langle \dfrac{y}{1+x^2} + \tan^{-1} z, \tan^{-1} x, \dfrac{x}{1+z^2} \right\rangle$

B 30. A force field $\mathbf{F}(x, y) = (3x^2 + 6xy^2)\mathbf{i} + (6x^2y + 4y^2)\mathbf{j}$ acts on an object moving in the plane. Show that $\mathbf{F}$ is conservative, and find a scalar potential for $\mathbf{F}$. How much work is done as the object moves from $(1, 0)$ to $(0, 1)$ along any path connecting these points?

Verify that each line integral in Problems 31–36 is independent of path and then find its value.

31. $\displaystyle\int_C [(3x^2 + 2x + y^2)\,dx + (2xy + y^3)\,dy]$, where C is any path from $(0, 0)$ to $(1, 1)$ $\frac{13}{4}$
32. $\displaystyle\int_C [(xy \cos xy + \sin xy)\,dx + (x^2 \cos xy)\,dy]$, where C is any path from $\left(0, \frac{\pi}{18}\right)$ to $\left(1, \frac{\pi}{6}\right)$ $\frac{1}{2}$
33. $\displaystyle\int_C [(y + x^2)\,dx + (x + y^2)\,dy]$, where C is any path from $(-1, -1)$ to $(0, 3)$ 8
34. $\displaystyle\int_C [(3x^2y + y^2)\,dx + (x^3 + 2xy)\,dy]$, where C is the path given parametrically by $\mathbf{R}(t) = t\,\mathbf{i} + (t^2 + t - 2)\mathbf{j}$ for $0 \le t \le 2$ 64
35. $\displaystyle\int_C [(\sin y)\,dx + (3 + x \cos y)\,dy]$, where C is the path given parametrically by $\mathbf{R}(t) = 2 \sin\left(\frac{\pi t}{2}\right) \cos(\pi t)\mathbf{i} + (\sin^{-1} t)\mathbf{j}$ for $0 \le t \le 1$.
36. $\displaystyle\int_C [(e^x \cos y)\,dx + (-e^x \sin y)\,dy]$, where C is the path given parametrically by $\mathbf{R}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ for $0 \le t \le \frac{\pi}{2}$.

Evaluate the line integrals given in Problems 37–41 using the fundamental theorem of line integrals.

37. $\displaystyle\int_C (y\mathbf{i} + x\mathbf{j}) \cdot d\mathbf{R}$, where C is any path from $(0, 0)$ to $(2, 4)$
38. $\displaystyle\int_C (xy^2\mathbf{i} + x^2y\mathbf{j}) \cdot d\mathbf{R}$, where C is any path from $(4, 1)$ to $(0, 0)$
39. $\displaystyle\int_C (2y\,dx + 2x\,dy)$, where C is the line segment from $(0, 0)$ to $(4, 4)$
40. $\displaystyle\int_C (e^x \sin y\,dx + e^x \cos y\,dy)$, where C is any smooth curve from $(0, 0)$ to $(0, 2\pi)$
41. $$\int_C \left\{ \left[\tan^{-1} \frac{y}{x} - \frac{xy}{x^2 + y^2} \right] dx + \left[\frac{x^2}{x^2 + y^2} + e^{-y}(1 - y) \right] dy \right\},$$
where C is any smooth curve from $(1, 1)$ to $(-1, 2)$
42. Find a function g so $g(x)\mathbf{F}(x, y)$ is conservative, where
$$\mathbf{F}(x, y) = (x^2 + y^2 + x)\mathbf{i} + xy\mathbf{j}$$
43. Find a function g so $g(x)\,\mathbf{F}(x, y)$ is conservative, where
$$\mathbf{F}(x, y) = (x^4 + y^4)\mathbf{i} - (xy^3)\mathbf{j}$$
44. **a.** Over what region in the xy-plane will the line integral
$$\int_C [(-yx^{-2} + x^{-1})\,dx + x^{-1}\,dy]$$
be independent of path?
b. Evaluate the line integral in part **a** if C is defined by
$$\mathbf{R}(t) = (\cos^3 t)\mathbf{i} + (\sin 3t)\mathbf{j}$$
for $0 \le t \le \frac{\pi}{3}$.
45. The **gravitational force field F** between two particles of masses M and m separated by a distance r is modeled by
$$\mathbf{F}(x, y, z) = -\frac{KmM}{r^3}\mathbf{R}$$
where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and K is the gravitational constant.
a. Show that $\mathbf{F}$ is conservative by finding a scalar potential for $\mathbf{F}$. The scalar potential function f is often called the **Newtonian potential.**
b. Compute the amount of work done against the force field $\mathbf{F}$ in moving an object from the point $P(a_1, b_1, c_1)$ to $Q(a_2, b_2, c_2)$.
46. Let $\mathbf{F}(x, y) = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$.
a. Compute the line integral $\int_{C_1} \mathbf{F} \cdot d\mathbf{R}$ where C_1 is the upper semicircle $y = \sqrt{1 - x^2}$ traversed counterclockwise. What is the value of $\int_{C_2} \mathbf{F} \cdot d\mathbf{R}$ if C_2 is the lower semicircle $y = -\sqrt{1 - x^2}$ also traversed counterclockwise?
b. Show that if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, then
$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right)$$
but $\mathbf{F}$ is not conservative on the unit disk $x^2 + y^2 \le 1$.
47. **Modeling Problem** A person whirls a bucket filled with water in a circle of radius 3 ft at the rate of 1 revolution per second. If the bucket and water weigh 30 lb, how much work is done by the force that keeps the bucket moving in a circular path?

In Problems 48 and 49, you are to experiment with the notion of computing work along a path, with and without the benefit of independence of path. Suppose you are to power a boat of some kind

from point $A(0, 0)$ to point $B(2, 1)$, and the primary consideration is the force of the wind, which generally opposes you. You are to investigate the effect of taking different paths from A to B.

48. **a.** Suppose the wind force is $\mathbf{F} = \langle -a, -b \rangle$, for a and b positive. Compute the work involved along the straight line path between A and B; then along a second path of your choice. Does the path matter here? Why or why not?

b. Due to the effect of the harbor you are entering, suppose the wind force is $\langle -a, -ae^{-y} \rangle$. Again, compute the work along two paths as in part **a**. Does the path matter here? Why or why not?

c. Repeat part **b** for $\langle -a, -ae^{-y+x/9} \rangle$.

49. **Exploration Problem** An important type of problem in several fields of application is, "Can we find the optimal path to minimize the work?" You are to explore this issue in regard to the wind force in Problem 48c

$$\mathbf{F} = \langle -a, -ae^{-y+x/9} \rangle$$

a. You should have the work computed for two different paths from the previous problem. By looking at these numbers and carefully studying $\mathbf{F}$, you should see that we will be rewarded (or punished) by changing the path slightly from the straight-line path. What do your observations suggest regarding trying to minimize the work?

b. Attempt to find an (approximate) optimal path, starting with your observations in part **a** and common sense. For example, the path could be described by a parabola (or higher-degree polynomial) or by some trigonometric function. Find a "good" path for this purpose.

c. Explore the following question: "Is there a realistic optimal path from A to B?" Consider two conditions: First, there is no restriction on the path; second, suppose there is a shore-line at $y = 1$ so that one's path cannot exceed this limit. *Hint*: One approach might be to consider all parabolic paths of the form $y = x(b - ax)$, where a and b are non-negative, and $y(2) = 1$. In this case, you can eliminate b and do a one-parameter study.

d. Assume the path in part **c** cannot go above $y = 1$. You may have discovered the optimal path. What is it? If you did the parabolic study suggested in part **c**, there is an optimal path in this case. What is it?

50. Show that if the vector field

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

51. Let f and g be differentiable functions of one variable. Show that the vector field $\mathbf{F} = [f(x) + y]\,\mathbf{i} + [g(y) + x]\,\mathbf{j}$ is conservative, and find the corresponding potential function.

52. Let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{R}\|$. Show that the work done in moving an object from a distance r_1 to a distance r_2 in the central force field $\mathbf{F} = \mathbf{R}/r^3$ is given by

$$W = \frac{1}{r_1} - \frac{1}{r_2}$$

53. An object of mass m moves along a trajectory $\mathbf{R}(t)$ with velocity $\mathbf{V}(t)$ in a conservative force field $\mathbf{F}$. Let $\mathbf{R}(t_0) = \mathbf{Q}_0$ and $\mathbf{R}(t_1) = \mathbf{Q}_1$ be the initial and terminal points on the trajectory.

a. Show that the work done on the object is

$W = K(t_1) - K(t_0)$, where $K(t) = \frac{1}{2}m\|\mathbf{V}(t)\|^2$ is the object's *kinetic energy*.

b. Let f be a scalar potential for $\mathbf{F}$. Then $P(t) = -f(t)$ is the *potential energy* of the object. Prove the *law of conservation of energy*—namely,

$$P(t_0) + K(t_0) = P(t_1) + K(t_1)$$

54. Complete the proof of Theorem 13.5 by showing that if $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path in the open connected set D, then $\mathbf{F}$ is conservative. Do this by completing the following steps:

a. Let $P(a, b)$ be a fixed point in D, and let $Q(x, y)$ be any point in D. Define the function $f(x, y)$ by

$$f(x, y) = \int_P^Q \mathbf{F} \cdot d\mathbf{R}$$

Let $Q_1(x, y)$ be a point in D other than Q that lies on the same horizontal line as Q. Let C_1 be any curve in D from P to Q_1, and let C_2 be the horizontal line segment joining Q_1 to Q.

Show that if $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$, then $\dfrac{\partial f}{\partial x} = M(x, y)$.

b. Using a similar argument (only with vertical line segments), show that $\dfrac{\partial f}{\partial y} = N(x, y)$. Conclude that since $\mathbf{F} = \nabla f$, $\mathbf{F}$ must be conservative in D.

13.4 Green's Theorem

IN THIS SECTION Green's theorem, area as a line integral, Green's theorem for multiply-connected regions, alternative forms of Green's theorem, normal derivatives

The fundamental theorem of calculus, $\displaystyle\int_a^b \frac{dF}{dx}\,dx = F(b) - F(a)$, can be described as saying that when the derivative $\dfrac{dF}{dx}$ is integrated over the closed interval $a \le x \le b$, the result is the same as that obtained by evaluating $F(x)$ at the "boundary points" a and b and forming the difference $F(b) - F(a)$. We obtained the analogous result

$$\int_C \nabla f \cdot d\mathbf{R} = f(Q) - f(P)$$

in Section 13.3, and our next goal is to obtain a different kind of analogue to the fundamental theorem of calculus called **Green's theorem** after the English mathematician George Green (Historical Quest in Problem 39).

GREEN'S THEOREM

Green's theorem relates a line integral around a closed curve to a double integral over the region contained by the curve. We begin with some terminology. A **Jordan curve**, named for the French mathematician Camille Jordan (1838–1922), is a closed curve C that does not intersect itself (see Figure 13.20). A simply connected region D in the plane has the property that it is connected and the interior of every Jordan curve C in D also lies in D, as shown in Figure 13.20.

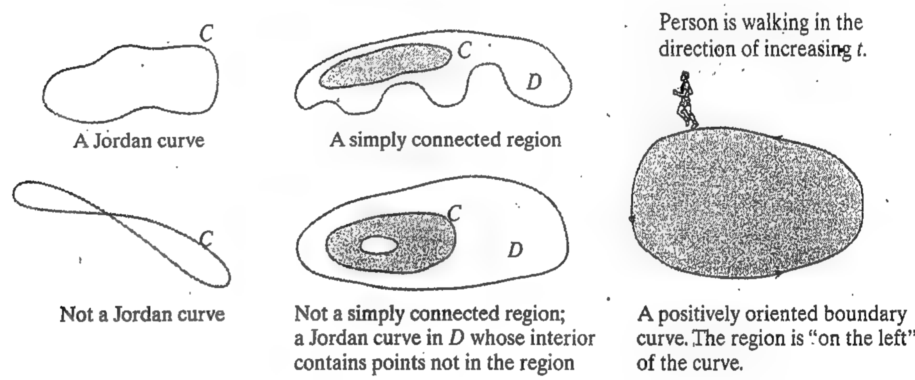


Figure 13.20 A Jordan curve is a closed curve with no self-intersections. A region is simply connected if every Jordan curve has all its interior points in D.

Picture yourself as a point moving along a curve. If the region D stays on your left as you, the point, move along the curve C with increasing t, then C is said to be **positively oriented** (see Figure 13.20). Now we are ready to state Green's theorem.

Graffiti on the wall of a high school playground in Ramat Gan (a suburb of Tel Aviv). *Courtesy of Eli Maor.*

THEOREM 13.6 **Green's theorem**

See HW copy for proof

Let D be a simply connected region that is bounded by the positively oriented piecewise smooth Jordan curve C. Then if the vector field $\mathbf{F}(x, y) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ is continuously differentiable on D, we have

$$\oint_C (M\,dx + N\,dy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

➡ **What This Says** This theorem expresses an important relationship between a line integral around a simple closed curve (a curve is **simple** if there are no self-intersections and **closed** if there are no endpoints) in the plane and a double integral over the region bounded by the curve. It is one of the most important and elegant theorems in calculus. Take special note that D is required to be simply connected with a *positively oriented* boundary C.

Proof A **standard region** is one in which no vertical or horizontal line can intersect the boundary curve more than twice (see Figure 13.21). We will prove Green's theorem

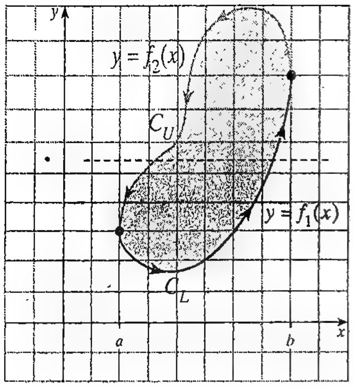


Figure 13.21 Standard region: No vertical or horizontal line intersects the boundary more than twice.

for the special case where D is a standard region, and then we will indicate how to extend the proof to more general regions. Suppose D is a standard region with boundary curve C. We begin by showing that

$$\iint_D \frac{\partial M}{\partial y}\,dx\,dy = -\int_C M\,dx$$

Because D is a standard region, as shown in Figure 13.21, the boundary curve C is composed of a lower portion C_L and an upper portion C_U, which are the graphs of functions $f_1(x)$ and $f_2(x)$, respectively, on a certain interval $a \le x \le b$. Then we can evaluate the double integral by iterated integration:

$$\begin{aligned}\iint_D \frac{\partial M}{\partial y}\,dx\,dy &= \iint_D \frac{\partial M}{\partial y}\,dy\,dx = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy\right] dx\\ &= \int_a^b M[x, f_2(x)]\,dx - \int_a^b M[x, f_1(x)]\,dx = \int_{-C_U} M\,dx - \int_{C_L} M\,dx\\ &= -\left[\int_{C_U} M\,dx + \int_{C_L} M\,dx\right] = -\oint_C M\,dx\end{aligned}$$

A similar argument shows that $\displaystyle\iint_D \frac{\partial N}{\partial x}\,dx\,dy = \oint_C N\,dy$. Thus,

$$\begin{aligned}\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA &= \iint_D \frac{\partial N}{\partial x}\,dx\,dy - \iint_D \frac{\partial M}{\partial y}\,dx\,dy\\ &= \oint_C N\,dy - \oint_C (-M)\,dx = \oint_C (M\,dx + N\,dy)\end{aligned}$$

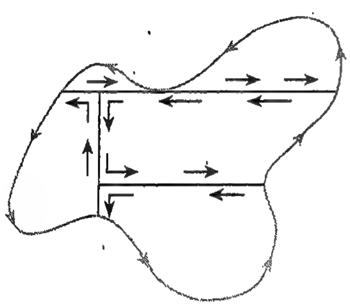


Figure 13.22 General case: The region is decomposed into a finite number of standard regions by cuts.

This completes the proof for a standard region. If D is not a standard region, it can be decomposed into a number of standard subregions by using horizontal and vertical "cuts," as shown in Figure 13.22. The proof for the standard region is then applied to each of these subregions, and the results are added. The line integrals along the cuts cancel in pairs, and after cancellation, the only remaining line integral is the one along the outer boundary C. Thus,

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

The case where there is one cut is considered in Problem 40. ❑

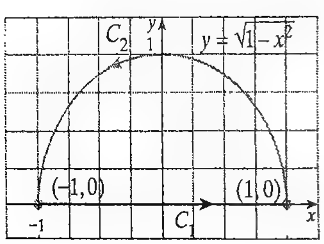


Figure 13.23 Path C

EXAMPLE 1 Using Green's theorem

Show that Green's theorem is true for the line integral $\displaystyle\oint_C (-y\,dx + x\,dy)$, where C is the closed path shown in Figure 13.23.

Solution

First, evaluate the line integral directly. The curve C consists of the line segment C_1 from $(-1, 0)$ to $(1, 0)$, followed by the semicircular arc C_2 from $(1, 0)$ back to $(-1, 0)$.

We parameterize each of these:

$$
\begin{array}{llll}
C_1: & x = t, & y = 0 & -1 \le t \le 1 \\
 & dx = dt, & dy = 0 & \\
C_2: & x = \cos u, & y = \sin u & 0 \le u \le \pi \\
 & dx = -\sin u\,du, & dy = \cos u\,du &
\end{array}
$$

$$
\begin{aligned}
\oint_C (-y\,dx + x\,dy) &= \int_{C_1} (-y\,dx + x\,dy) + \int_{C_2} (-y\,dx + x\,dy) \\
&= \int_{-1}^{1} [-0\,dt + t \cdot 0] + \int_0^{\pi} [-\sin u(-\sin u\,du) + \cos u(\cos u\,du)] \\
&= \int_0^{\pi} (\sin^2 u + \cos^2 u)\,du = \int_0^{\pi} 1\,du = \pi
\end{aligned}
$$

Next, we use Green's theorem to evaluate this integral. Note that the boundary C is a Jordan curve and $M = -y$, $N = x$, so that $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ is continuously differentiable. We now apply Green's theorem:

$$
\begin{aligned}
\oint_C (-y\,dx + x\,dy) &= \iint_D \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right) dA = \iint_D 2\,dA \\
&= 2(\text{AREA OF SEMICIRCLE}) = 2\left[\tfrac{1}{2}\pi(1)^2\right] = \pi
\end{aligned}
$$

■

EXAMPLE 2 Computing work with Green's theorem

A closed path C in the plane is defined by Figure 13.24. Find the work done by an object moving along C in the force field $\mathbf{F}(x, y) = (x + xy^2)\,\mathbf{i} + 2(x^2y - y^2 \sin y)\mathbf{j}$.

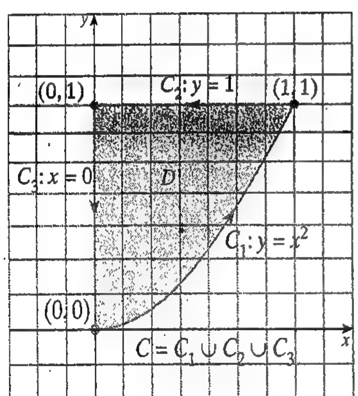


Figure 13.24 Closed path C

Solution

The work done, W, is given by the line integral $\oint_C \mathbf{F} \cdot d\mathbf{R}$. Note that $\mathbf{F}$ is continuously differentiable on the region D enclosed by C, and since D is simply connected with a positively oriented boundary (namely, C), the hypotheses of Green's theorem are satisfied. We find that

$$
\begin{aligned}
W = \oint_C \mathbf{F} \cdot d\mathbf{R} &= \iint_D \left[\frac{\partial}{\partial x}(2x^2y - 2y^2 \sin y) - \frac{\partial}{\partial y}(x + xy^2) \right] dA \\
&= \iint_D (4xy - 2xy)\,dA = 2\int_0^1 \int_{x^2}^1 xy\,dy\,dx = 2\int_0^1 \tfrac{1}{2}xy^2\Big|_{y=x^2}^{y=1} dx \\
&= \int_0^1 (x - x^5)\,dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{6}x^6\right]\Big|_0^1 = \tfrac{1}{3}
\end{aligned}
$$

■

AREA AS A LINE INTEGRAL

A line integral can be used to compute an area of a region in the plane by applying the following theorem.

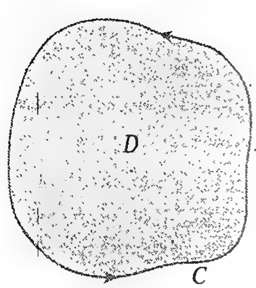


Figure 13.25 Area of region D

THEOREM 13.7 **Area as a line integral**

Let D be a simply connected region in the plane with piecewise smooth, positively oriented closed boundary C, as shown in Figure 13.25. Then the area A of region D is given by each of the following line integrals:

$$
A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C [x\,dy - y\,dx]
$$

Proof We will prove that

$$A = \frac{1}{2}\oint_C [x\,dy - y\,dx]$$

and leave the other two area formulas for the reader (see Problem 44).

Let $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$. Then since $\mathbf{F}$ is continuously differentiable on D, Green's theorem applies. We have

$$\oint_C (-y\,dx + x\,dy) = \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right] dA = \iint_D 2\,dA = 2A$$

so that $A = \dfrac{1}{2}\oint_C (-y\,dx + x\,dy)$. ☐

WARNING This gives us yet another technique for finding the area of a region, especially when its boundary is specified in parametric form. In finding area, the function $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ does not change. Do not forget the factor one-half after you finish the integration.

EXAMPLE 3 Area enclosed by an ellipse

Show that the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has area $\pi\, ab$.

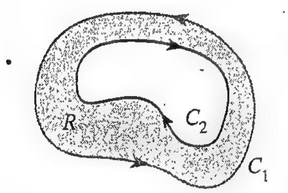
Solution

The elliptical path E is given parametrically by $x = a\cos\theta$, $y = b\sin\theta$ for $0 \le \theta \le 2\pi$. We find $dx = -a\sin\theta\, d\theta$, $dy = b\cos\theta\, d\theta$. If A is the area of this ellipse, then

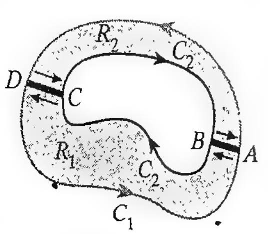
$$\begin{aligned}
A &= \tfrac{1}{2}\oint_C (-y\,dx + x\,dy)\\
&= \tfrac{1}{2}\int_0^{2\pi} [-(b\sin\theta)(-a\sin\theta\,d\theta) + (a\cos\theta)(b\cos\theta\,d\theta)]\\
&= \tfrac{1}{2}\int_0^{2\pi} ab(\sin^2\theta + \cos^2\theta)\,d\theta\\
&= \tfrac{1}{2}\int_0^{2\pi} ab\,d\theta\\
&= \tfrac{1}{2}ab(2\pi - 0)\\
&= \pi ab
\end{aligned}$$

■

GREEN'S THEOREM FOR MULTIPLY-CONNECTED REGIONS



a. A doubly-connected region with oriented boundary curves C_1 and C_2.



b. Two "cuts" are made through the hole.

Figure 13.26 Multiply-connected region

In the statement of Green's theorem, we require the region R inside the boundary curve C to be simply connected, but the theorem can be extended to multiply-connected regions, that is, regions with one or more "holes." A region with a single hole is shown in Figure 13.26a. The boundary of this region consists of an "outer" curve C_1 and an "inner" curve C_2, oriented so that the region R is always on the left as we travel around the boundary, which means that C_1 is oriented counterclockwise and C_2 clockwise.

We now make cuts AB and CD through the region to the hole, as indicated in Figure 13.26b. Let R_1 be the simply connected region contained by the closed curve C_3 that begins at A, extends along the cut to B, and then clockwise along the bottom of the curve C_2 to C, along the cut to D, and counterclockwise along the bottom of C_1 back to A. Similarly, let R_2 be the region contained by the curve C_4 that begins at D and extends to C along the cut, to B along the top of C_2, to A along the second cut, and back to D along the top part of C_1. Then, if the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is continuously differentiable on R, we can apply Green's theorem to show

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \oint_{C_3} (M\,dx + N\,dy) + \oint_{C_4} (M\,dx + N\,dy)$$

But the line integrals from A to B and C to D cancel those from B to A and D to C, leaving only the line integrals along the original boundary curves C_1 and C_2. Thus, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_{C_1} (M\,dx + N\,dy) + \oint_{C_2} (M\,dx + N\,dy)$$

Green's Theorem for Doubly-Connected Regions

Let R be a doubly-connected region (one hole) in the plane, with outer boundary C_1 oriented counterclockwise and boundary C_2 of the hole oriented clockwise. If the boundary curves and $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ satisfy the hypotheses of Green's theorem, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_{C_1} (M\,dx + N\,dy) + \oint_{C_2} (M\,dx + N\,dy)$$

Example 4 illustrates one way this result can be used.

do imp **EXAMPLE 4 Green's theorem for a region containing a singular point**

Show that $\displaystyle\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = 2\pi$, where C is any piecewise smooth Jordan curve enclosing the origin (0, 0).

Solution

Let $M(x, y) = \dfrac{-y}{x^2 + y^2}$ and $N = \dfrac{x}{x^2 + y^2}$. Then

$$\frac{\partial N}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial M}{\partial y}$$

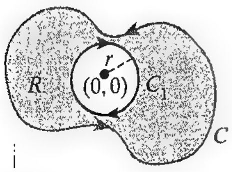


Figure 13.27 The region R for doubly-connected regions

at any point (x, y) other than the origin. Next, let C_1 be a circle centered at the origin with radius r so small that the entire circle is contained in C, and let R be the region between the curve C and the circle C_1, as shown in Figure 13.27.

We know that $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$ throughout R (since R does not contain the origin) and Green's theorem for doubly-connected regions tells us that

$$\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} + \oint_{C_1} \frac{-y\,dx + x\,dy}{x^2 + y^2} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0$$

so that

$$\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = -\oint_{C_1} \frac{-y\,dx + x\,dy}{x^2 + y^2} = \oint_{-C_1} \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

where $-C_1$ is the circle C_1 traversed counterclockwise instead of clockwise. In other words, we can find the value of the given line integral about the curve C by finding the line integral about the circle $-C_1$. To do this, we parameterize $-C_1$ by

$$x = r\cos\theta, \qquad y = r\sin\theta \qquad \text{for } 0 \le \theta \le 2\pi$$

and find that

$$\begin{aligned}\oint_{-C_1} \frac{-y\,dx + x\,dy}{x^2+y^2} &= \int_0^{2\pi} \frac{-r\sin\theta(-r\sin\theta\,d\theta) + r\cos\theta(r\cos\theta\,d\theta)}{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \int_0^{2\pi} \frac{r^2(\sin^2\theta + \cos^2\theta)}{r^2(\sin^2\theta+\cos^2\theta)}\,d\theta = \int_0^{2\pi} 1\,d\theta = 2\pi\end{aligned}$$

Thus,

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \oint_{-C_1} \frac{-y\,dx + x\,dy}{x^2+y^2} = 2\pi \qquad \blacksquare$$

Note that the line integral in Example 4 is not independent of path in R because there are closed paths (like C_1) along which the line integral is not 0.

ALTERNATIVE FORMS OF GREEN'S THEOREM

Green's theorem can be expressed in two forms that generalize nicely to $\mathbb{R}^3$. For the first, note that the curl of the vector field $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is given by

$$\begin{aligned}\text{curl}\,\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M(x,y) & N(x,y) & 0 \end{vmatrix} \\ &= 0\mathbf{i} + 0\mathbf{j} + \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]\mathbf{k} \\ &= \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]\mathbf{k} \qquad \textit{Because } M \textit{ and } N \textit{ are functions of only } x \textit{ and } y\end{aligned}$$

so

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_D (\text{curl}\,\mathbf{F}\cdot\mathbf{k})\,dA$$

On the other hand, the line integral in Green's theorem can be expressed as

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_C \mathbf{F}\cdot\frac{d\mathbf{R}}{ds}\,ds = \oint_C \mathbf{F}\cdot\mathbf{T}\,ds$$

By combining these results, we obtain

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \qquad \textit{Green's theorem}$$

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_D (\text{curl}\,\mathbf{F}\cdot\mathbf{k})\,dA$$

When we extend this result to surfaces in $\mathbb{R}^3$ it will be called *Stokes' theorem*. We will examine Stokes' theorem in Section 13.6 and show how it can be interpreted in terms of the circulation of a fluid flow.

We have just seen how Green's theorem can be written in terms of $\mathbf{F} \cdot \mathbf{T}$, the tangential component of the vector field $\mathbf{F}$. The following example shows how Green's theorem can also be expressed in terms of $\mathbf{F} \cdot \mathbf{N}$, the normal component of $\mathbf{F}$. When this result is extended to $\mathbb{R}^3$ in Section 13.7, it will be called the *divergence theorem.*

EXAMPLE 5 The divergence theorem in the plane

Suppose $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ with a piecewise smooth boundary C. Show that

$$\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \operatorname{div} \mathbf{F}\, dA$$

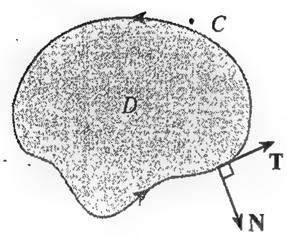


Figure 13.28 The outward unit normal to $\mathbf{R}(s) = \langle x(s), y(s) \rangle$ is $\mathbf{N} = \langle y'(s), -x'(s) \rangle$

Solution

Parameterize C with the arc length parameter s, so that $\mathbf{R}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$ is the position vector on C. A unit tangent vector $\mathbf{T}$ to the curve C is $\mathbf{T} = \mathbf{R}'(s) = x'(s)\mathbf{i} + y'(s)\mathbf{j}$, which means that an outward normal vector is $\mathbf{N} = y'(s)\ \mathbf{i} - x'(s)\mathbf{j}$ (see Figure 13.28).

We now apply Green's theorem to find a representation using div $\mathbf{F}$.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{N}\, ds &= \int_a^b (M\mathbf{i} + N\mathbf{j}) \cdot [y'(s)\mathbf{i} - x'(s)\mathbf{j}]\, ds \\
&= \int_a^b \left(M\frac{dy}{ds} - N\frac{dx}{ds} \right) ds \\
&= \oint_C (-N\, dx + M\, dy) \\
&= \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy \qquad \textit{Green's theorem} \\
&= \iint_D \operatorname{div} \mathbf{F}\, dA
\end{aligned}$$

■

We list these alternative forms of Green's theorem for easy reference.

Alternative Forms of Green's Theorem

Let D be a simply connected region with a positively oriented boundary C. Then if the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is continuously differentiable on D, we have

$$\oint_C \underbrace{\mathbf{F} \cdot d\mathbf{R}}_{\text{Tangential component of } \mathbf{F}} = \oint_C (M\, dx + N\, dy)$$

$$= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_D (\operatorname{curl} \mathbf{F} \cdot \mathbf{k})\, dA$$

$$\oint_C \underbrace{\mathbf{F} \cdot \mathbf{N}}_{\text{Normal component of } \mathbf{F}} ds = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA = \iint_D \operatorname{div} \mathbf{F}\, dA$$

When we extend this result to surfaces in $\mathbb{R}^3$ it will be called *Stokes' theorem*. We will examine Stokes' theorem in Section 13.6 and show how it can be interpreted in terms of the circulation of a fluid flow.

We have just seen how Green's theorem can be written in terms of $\mathbf{F}\cdot\mathbf{T}$, the tangential component of the vector field $\mathbf{F}$. The following example shows how Green's theorem can also be expressed in terms of $\mathbf{F}\cdot\mathbf{N}$, the normal component of $\mathbf{F}$. When this result is extended to $\mathbb{R}^3$ in Section 13.7, it will be called the *divergence theorem*.

EXAMPLE 5 The divergence theorem in the plane

Suppose $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ with a piecewise smooth boundary C. Show that

$$\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \iint_D \operatorname{div}\mathbf{F}\,dA$$

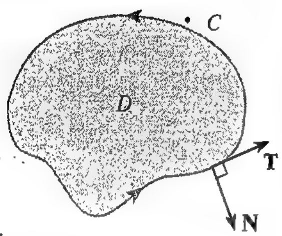


Figure 13.28 The outward unit normal to $\mathbf{R}(s) = \langle x(s), y(s)\rangle$ is $\mathbf{N} = \langle y'(s), -x'(s)\rangle$

Solution

Parameterize C with the arc length parameter s, so that $\mathbf{R}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$ is the position vector on C. A unit tangent vector $\mathbf{T}$ to the curve C is $\mathbf{T} = \mathbf{R}'(s) = x'(s)\mathbf{i} + y'(s)\mathbf{j}$, which means that an outward normal vector is $\mathbf{N} = y'(s)\ \mathbf{i} - x'(s)\mathbf{j}$ (see Figure 13.28).

We now apply Green's theorem to find a representation using div $\mathbf{F}$.

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{N}\,ds &= \int_a^b (M\mathbf{i} + N\mathbf{j})\cdot[y'(s)\mathbf{i} - x'(s)\mathbf{j}]\,ds \\
&= \int_a^b \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds \\
&= \oint_C (-N\,dx + M\,dy) \\
&= \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \qquad \text{Green's theorem} \\
&= \iint_D \operatorname{div}\mathbf{F}\,dA
\end{aligned}$$

■

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Alternative Forms of Green's Theorem

Let D be a simply connected region with a positively oriented boundary C. Then if the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is continuously differentiable on D, we have

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$$= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_D (\operatorname{curl}\mathbf{F}\cdot\mathbf{k})\,dA$$

$$\oint_C \underbrace{\mathbf{F}\cdot\mathbf{N}}_{\text{Normal component of }\mathbf{F}}\,ds = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_D \operatorname{div}\mathbf{F}\,dA$$

NORMAL DERIVATIVES

In physics, some important applications of Green's theorem involve the *normal derivative* of a scalar function f, which is defined as the directional derivative of f in the direction of the outward normal vector $\mathbf{N}$ to some curve or surface.

Normal Derivative

The **normal derivative** of f, denoted by $\partial f/\partial n$, is the directional derivative of f in the direction of the normal vector pointing to the exterior of the domain of f. In other words,

$$\frac{\partial f}{\partial n} = \nabla f \cdot \mathbf{N}$$

where $\mathbf{N}$ is the outward unit normal vector.

The following example illustrates how Green's theorem can be used in connection with the normal derivative. Additional examples are found in the problem set.

EXAMPLE 6 Green's formula for the integral of the Laplacian

Suppose f is a scalar function with continuous first and second partial derivatives in the simply connected region D. If the piecewise smooth closed curve C bounds D, show that

$$\iint_D \nabla^2 f\, dx\, dy = \oint_C \frac{\partial f}{\partial n}\, ds$$

where $\nabla^2 f = f_{xx} + f_{yy}$ is the Laplacian of f and $\dfrac{\partial f}{\partial n} = \nabla f \cdot \mathbf{N}$ is the normal derivative vector.

Solution

Let $u = -\dfrac{\partial f}{\partial y}$ and $v = \dfrac{\partial f}{\partial x}$. Then we have

$$\nabla^2 f = f_{xx} + f_{yy} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\iint_D \nabla^2 f\, dx\, dy = \iint_D \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$$

$$= \oint_C (u\, dx + v\, dy) \qquad \text{Green's theorem}$$

$$= \oint_C \left(u\frac{dx}{ds} + v\frac{dy}{ds}\right) ds$$

$$= \oint_C \left(-\frac{\partial f}{\partial y}\frac{dx}{ds} + \frac{\partial f}{\partial x}\frac{dy}{ds}\right) ds$$

$$= \oint_C \left(f_x\frac{dy}{ds} - f_y\frac{dx}{ds}\right) ds$$

$$= \oint_C \nabla f \cdot \left(\frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}\right) ds$$

$$= \oint_C \nabla f \cdot \mathbf{N}\, ds \qquad \text{Where } \mathbf{N} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j} \text{ is the outward unit normal vector to } C$$

$$= \oint_C \frac{\partial f}{\partial n}\, ds$$

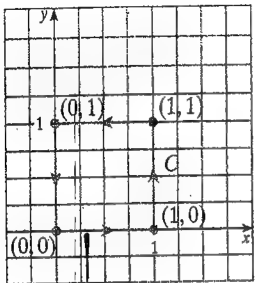
■

Guidelines (pg 60-81)

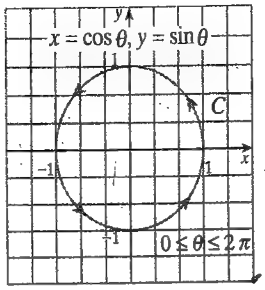
13.4 PROBLEM SET

A *In Problems 1–6, use Green's theorem to evaluate the given line integral around the indicated closed curve C. Then check your answer by parameterizing C.*

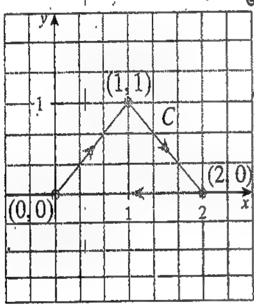
1. $\oint_C (y^2\,dx + x^2\,dy)$ 0



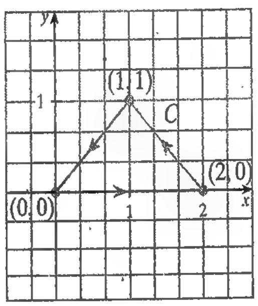
2. $\oint_C (y^3\,dx - x^3\,dy)$



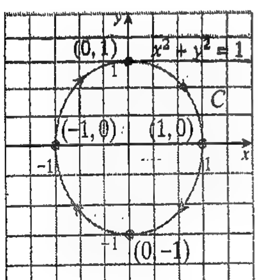
3. $\oint_C [(2x^2 + 3y)\,dx - 3y^2\,dy]$ 3



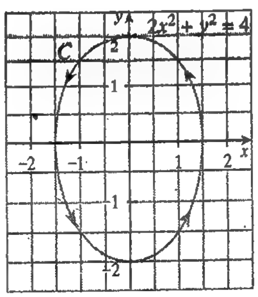
4. $\oint_C (y^2\,dx + 3xy^2\,dy)$



5. $\oint_C 4xy\,dx$ 0

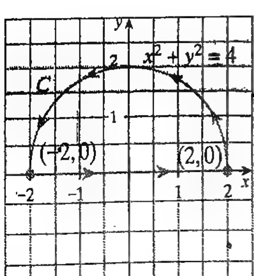


6. $\oint_C (4y\,dx - 3x\,dy)$

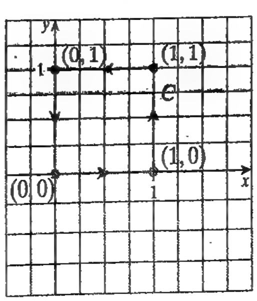


In Problems 7–14, use Green's theorem to evaluate the given line integral around the indicated closed curve C.

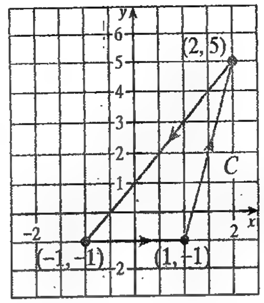
7. $\oint_C (2y\,dx - x\,dy)$



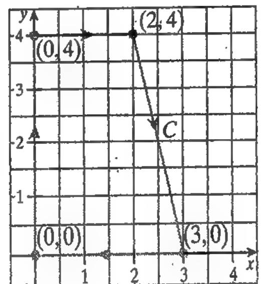
8. $\oint_C (e^x\,dx - \sin x\,dy)$



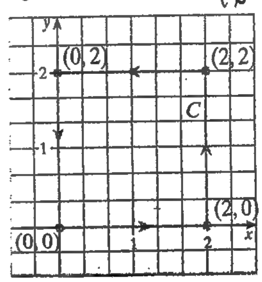
9. $\oint_C (x \sin x\,dx - e^{y^2}\,dy)$ 0



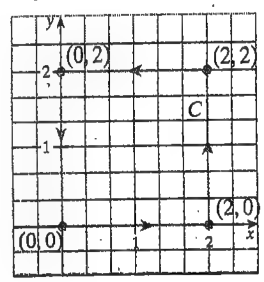
10. $\oint_C [(x + y)\,dx - (3x - 2y)\,dy]$ 40



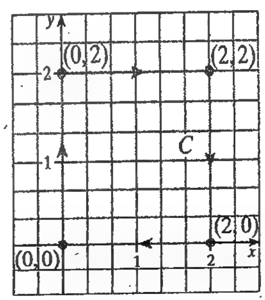
11. $\oint_C [(x - y^2)\,dx + 2xy\,dy]$ 16



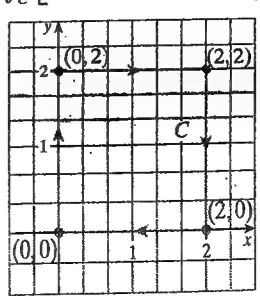
12. $\oint_C (y^2\,dx + x\,dy)$



13. $\oint_C [\sin x \cos y\,dx + \cos x \sin y\,dy]$ 0



14. $\oint_C \left[2x \tan^{-1} y\,dx - \dfrac{x^2y^2}{1 + y^2}\,dy\right]$ 8



15. Use Green's theorem to find the work done by the force field

$$\mathbf{F}(x, y) = (3y - 4x)\mathbf{i} + (4x - y)\mathbf{j}$$

when an object moves once counterclockwise around the ellipse $4x^2 + y^2 = 4$. 2π

16. Find the work done when an object moves in the force field $\mathbf{F}(x, y) = y^2\mathbf{i} + x^2\mathbf{j}$ counterclockwise around the circular path $x^2 + y^2 = 2$. 0

Use Theorem 13.7 to find the area enclosed by the regions described in Problems 17–20, and then check by using an appropriate formula.

17. circle $x^2 + y^2 = 4$ 4π

18. triangle with vertices (0, 0), (1, 1), and (0, 2)

19. trapezoid with vertices (0, 0), (4, 0), (1, 3), and (0, 3)

20. semicircle $y = \sqrt{4 - x^2}$ 2π

B 21. Evaluate the line integral $\oint_C (x^2y\,dx - y^2x\,dy)$, where C is the boundary of the region between the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$, traversed counterclockwise (including the x-axis). $\frac{\pi a^4}{4}$

22. Evaluate the line integral

$$\oint_C (3y\,dx - 2x\,dy)$$

where C is the cardioid $r = 1 + \sin\theta$, traversed counterclockwise.

23. Show that

$$\oint_C [(5 - xy - y^2)\,dx - (2xy - x^2)\,dy] = 3\overline{x}$$

where C is the square $0 \le x \le 1$, $0 \le y \le 1$ traversed counterclockwise and $\overline{x}$ is the x-coordinate of the centroid of the square.

24. Find the work done by the force field $\mathbf{F}(x, y) = (x + 2y^2)\mathbf{j}$ as an object moves once counterclockwise about the circle $(x - 2)^2 + y^2 = 1$.

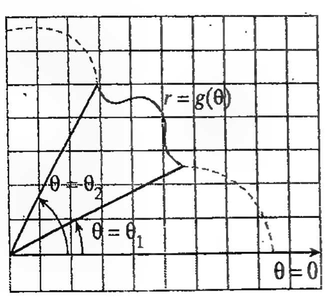
25. Let D be the region bounded by the Jordan curve C, and let A be the area of D. If $(\overline{x}, \overline{y})$ is the centroid of D, show that

$$\overline{x} = \frac{1}{2A}\oint_C x^2\,dy \quad \text{and} \quad \overline{y} = -\frac{1}{2A}\oint_C y^2\,dx$$

where C is traversed counterclockwise.

26. Use Theorem 13.7 and the polar transformation formulas $x = r\cos\theta$, $y = r\sin\theta$ to obtain the area formula in polar coordinates, namely,

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} [g(\theta)]^2\,d\theta$$



27. Evaluate $\oint_C \left[\left(\frac{-y}{x^2} + \frac{1}{x}\right)dx + \frac{1}{x}\,dy\right]$, where C is any Jordan curve that does not intersect the y-axis, traversed counterclockwise.

28. Evaluate $\oint_C \frac{x\,dx + y\,dy}{x^2 + y^2}$, where C is any Jordan curve whose interior does not contain the origin, traversed counterclockwise.

29. Evaluate the line integral

$$\oint_C \frac{x\,dx + y\,dy}{x^2 + y^2}$$

where C is any piecewise smooth Jordan curve enclosing the origin, traversed counterclockwise.

30. Evaluate $\oint_C \frac{-y\,dx + (x-1)\,dy}{(x-1)^2 + y^2}$, where C is any Jordan curve whose interior does not contain the point (1, 0), traversed counterclockwise.

31. Evaluate $\oint_C \frac{-(y+2)\,dx + (x-1)\,dy}{(x-1)^2 + (y+2)^2}$, where C is any Jordan curve whose interior does not contain the point (1, −2), traversed counterclockwise.

32. Evaluate $\oint_C \frac{\partial z}{\partial n}\,ds$, where $z(x, y) = 2x^2 + 3y^2$, and C is the circular path $x^2 + y^2 = 16$, traversed counterclockwise.

33. Evaluate $\oint_C \frac{\partial f}{\partial n}\,ds$, where $f(x, y) = x^2y - 2xy + y^2$, and C is the boundary of the unit square $0 \le x \le 1$, $0 \le y \le 1$, traversed counterclockwise.

34. Evaluate $\oint_C x\frac{\partial x}{\partial n}\,ds$, where C is the boundary of the unit square $0 \le x \le 1$, $0 \le y \le 1$, traversed counterclockwise.

35. If C is a Jordan curve, show that

$$\oint_C [(x - 3y)\,dx + (2x - y^2)\,dy] = 5A$$

where A is the area of the region D enclosed by C.

36. Use line integration to find the area of the region bounded by the curve C: $x = \cos^3 t$, $y = \sin^3 t$ for $0 \le t \le 2\pi$.

C 37. Prove the following theoretical application of Green's theorem: Let $\mathbf{F}(x, y) = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$ be continuously differentiable on the simply connected region D. Then $\mathbf{F}$ is conservative if and only if

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

throughout D.

38. Recall that a scalar function f with continuous first and second partial derivatives is said to be *harmonic* in a region D if $\nabla^2 f = 0$ (that is, if $f_{xx} + f_{yy} = 0$). If f is such a function and D is a simply connected region enclosed by the Jordan curve C, show that

$$\iint_D (f_x^2 + f_y^2)\,dx\,dy = \oint_C f\frac{\partial f}{\partial n}\,ds$$

39. **HISTORICAL QUEST** George Green (1793–1841) was the son of a baker who worked in his father's mill and studied mathematics and physics in his spare time, using only books he obtained from the library. In 1828, he published a memoir titled "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," which contains the result that now bears his name. Very few copies of the essay were printed and distributed, so few people knew of Green's results. In 1833, at the age of 40, he entered Cambridge University and graduated just four years before his death. His 1828 paper was discovered and publicized in 1845 by Sir William Thompson (later known as Lord Kelvin, 1824–1907), and Green finally received proper credit for his work. ■

In this Quest, use Green's theorem to prove the following two important results, known as **Green's formulas**. In both cases, D is a simply connected region enclosed by the Jordan curve C.

a. **Green's first formula.**

$$\iint_D [f\nabla^2 g + \nabla f \cdot \nabla g]\, dx\, dy = \oint_C f \frac{\partial g}{\partial n}\, ds$$

b. Once again start with the line integral and derive what is known as **Green's second formula.**

$$\iint_D [f\nabla^2 g - g\nabla^2 f]\, dx\, dy = \oint_C \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) ds$$

40. **HISTORICAL QUEST** The result known as "Green's theorem" in the West is called "Ostrogradsky's theorem" in Russia, after Mikhail Ostrogradsky. Although Ostrogradsky published over 80 papers during a successful career as a mathematician, today he is known, even in his homeland, only for his version of Green's theorem, which appeared as part of a series of results presented to the Academy of Sciences in 1828. ■

MIKHAIL OSTROGRADSKY
1801–1862

In this Quest you are asked to prove Green–Ostrogradsky's theorem in the plane for the nonstandard region D shown in Figure 13.29.

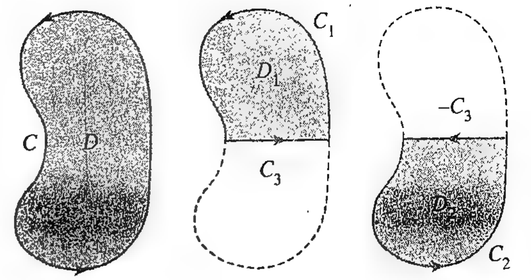


Figure 13.29 Green's theorem cut by a line L

Specifically, suppose that the line C_3 "cuts" the region D into two standard subregions D_1 and D_2. Apply Green's theorem to D_1 and D_2, and then combine the results to show that the theorem applies to the non-standard region D. *Hint*: The key is what happens along the "cut line" C_3.

41. Extend Green's theorem to a "triply-connected" region (two holes), such as the one shown in Figure 13.30.

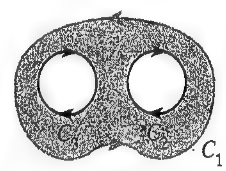


Figure 13.30 A triply-connected region

42. Suppose $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is continuously differentiable in a doubly-connected region R and that

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

throughout R. How many distinct values of I are there for the integral

$$I = \oint_C [M(x, y)\, dx + N(x, y)\, dy]$$

where C is a piecewise smooth Jordan curve in R?

43. Answer the question in Problem 42 for the case where R is triply-connected (two holes). See Figure 13.30.

44. Let D be a simply connected region in the plane with a piecewise smooth closed boundary C. Complete the proof of Theorem 13.7 by showing that the area of D is given by

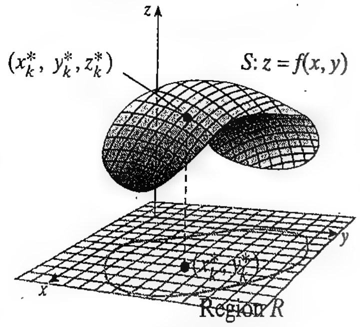
$$A = \oint_C x\, dy \quad \text{and} \quad A = -\oint_C y\, dx$$

13.5 Surface Integrals

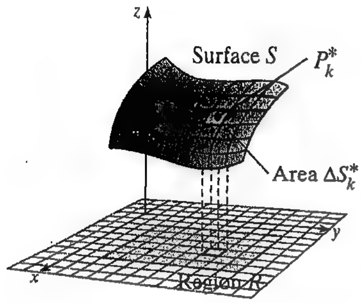
IN THIS SECTION surface integration, flux integrals, integrals over parametrically defined surfaces

SURFACE INTEGRATION

A surface is said to be smooth if there is a nonzero normal vector at each of its points, and it is piecewise smooth if it is composed of a finite number of smooth pieces.



a. A piecewise smooth surface S



b. Construction of a Riemann sum

Figure 13.31 Definition of a surface integral

A *surface integral* is a generalization of the line integral in which the integration is over a surface in space rather than a curve.

We begin by defining the surface integral of a continuous scalar function $g(x, y, z)$ over a piecewise smooth surface S, as shown in Figure 13.31a. Partition S into n subregions, the kth of which has area ΔS_k, and let $P_k^*(x_k^*, y_k^*, z_k^*)$ be a point chosen arbitrarily from the kth subregion, for $k = 1, 2, \ldots, n$ (see Figure 13.31b). Form the Riemann sum

$$\sum_{k=1}^{n} g(P_k^*)\Delta S_k$$

and take the limit as the largest of the ΔS_k tends to 0. If this limit exists, it is called the **surface integral of g over S** and is denoted by

$$\iint_S g(x, y, z)\, dS$$

Recall from Section 12.4 that when the surface S projects onto the region R in the xy-plane and S has the representation $z = f(x, y)$, then $dS = \sqrt{f_x^2 + f_y^2 + 1}\, dA$, where dA is either $dx\,dy$ or $dy\,dx$. We can now state the formula for evaluating a surface integral.

Surface Integral

Let S be a surface defined by $z = f(x, y)$ and R be its projection on the xy-plane. If f, f_x, and f_y are continuous in R and g is a continuous function of three variables on S, then the **surface integral** of g over S is

$$\iint_S g(x, y, z)\,dS = \iint_R g(x, y, f(x, y))\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

EXAMPLE 1 Evaluating a surface integral

Evaluate the surface integral

$$\iint_S g\, dS$$

shown in Figure 13.32, where $g(x, y, z) = xz + 2x^2 - 3xy$ and S is that portion of the plane $2x - 3y + z = 6$ that lies over the unit square R: $2 \le x \le 3$, $2 \le y \le 3$.

Solution

First, note that the equation of the plane can be written as $z = f(x, y)$ where $f(x, y) = 6 - 2x + 3y$. We have $f_x(x, y) = -2$ and $f_y(x, y) = 3$, so that

$$dS = \sqrt{f_x^2 + f_y^2 + 1}\, dA = \sqrt{(-2)^2 + (3)^2 + 1}\, dA = \sqrt{14}\, dA$$

Consequently,

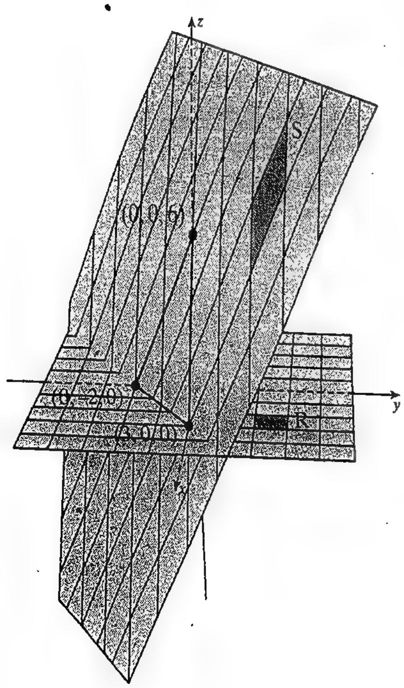


Figure 13.32 The portion of the plane that lies above the unit square R

$$\iint_S g\,dS = \iint_R (xz + 2x^2 - 3xy)\sqrt{14}\,dA$$

$$= \iint_R [x(6 - 2x + 3y) + 2x^2 - 3xy]\sqrt{14}\,dy\,dx \quad \text{Since } z = 6 - 2x + 3y$$

$$= \sqrt{14} \iint_R 6x\,dy\,dx$$

$$= 6\sqrt{14} \int_2^3 \int_2^3 x\,dy\,dx = 6\sqrt{14} \int_2^3 x\,dx = 15\sqrt{14} \quad \blacksquare$$

If the function g defined on S is simply $g(x, y, z) = 1$, then the surface integral gives the surface area of S.

Surface Area Formula

If S is a piecewise smooth surface, its area is given by

$$A = \iint_S dS$$

A useful application of surface integrals is to find the center of mass of a thin curved lamina whose shape is part of a given surface S, as shown in Figure 13.33.

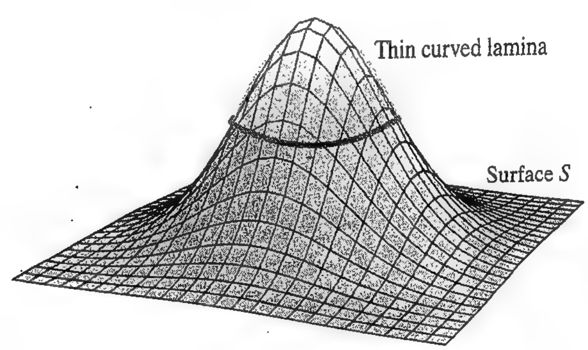


Figure 13.33 A thin lamina whose shape is the surface S

If $\delta(x, y, z)$ is the density (mass per unit area) at each point (x, y, z) on the lamina, then the *total mass*, m, of the lamina is given by the surface integral

MASS OF A LAMINA

$$m = \iint_S \delta(x, y, z)\,dS$$

and the center of mass of the surface is the point $C(\overline{x}, \overline{y}, \overline{z})$, where

CENTER OF MASS

$$\overline{x} = \frac{1}{m}\iint_S x\delta\,dS, \qquad \overline{y} = \frac{1}{m}\iint_S y\delta\,dS, \qquad \overline{z} = \frac{1}{m}\iint_S z\delta\,dS$$

These formulas may be derived by essentially the same approach used in our previous work with moments and centroids of solid regions. (See, for example, Sections 6.5 and 12.6.)

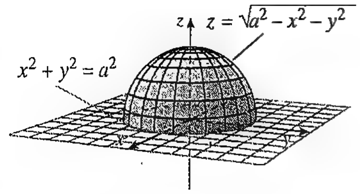


Figure 13.34 The hemisphere $z = \sqrt{a^2 - x^2 - y^2}$

EXAMPLE 2 Mass of a curved lamina

Find the mass of a lamina of density $\delta(x, y, z) = z$ in the shape of the hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$, as shown in Figure 13.34.

Solution

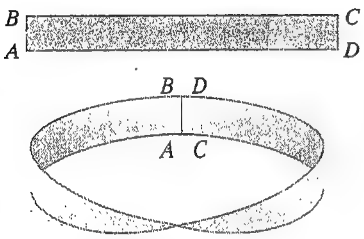
We begin by calculating dS.

$$z_x = \tfrac{1}{2}(a^2 - x^2 - y^2)^{-1/2}(-2x) = -x(a^2 - x^2 - y^2)^{-1/2}$$
$$z_y = \tfrac{1}{2}(a^2 - x^2 - y^2)^{-1/2}(-2y) = -y(a^2 - x^2 - y^2)^{-1/2}$$
$$dS = \sqrt{z_x^2 + z_y^2 + 1}\,dA = \sqrt{\frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1}\,dA$$
$$= \sqrt{\frac{a^2}{a^2 - x^2 - y^2}}\,dA = a(a^2 - x^2 - y^2)^{-1/2}\,dA$$

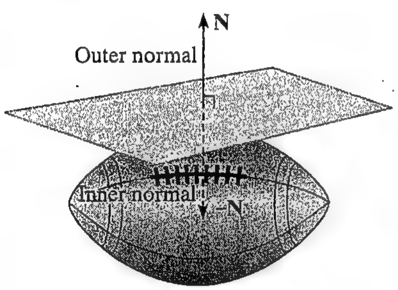
The surface projects onto the disk $x^2 + y^2 \le a^2$ in the xy-plane, so the mass of the hemisphere S is given by

$$m = \iint_S \delta(x, y, z)\,dS = \iint_S z\,dS$$
$$= \iint_R (a^2 - x^2 - y^2)^{1/2} a(a^2 - x^2 - y^2)^{-1/2}\,dA$$
$$= a\iint_R dA = a(\pi a^2) = \pi a^3 \quad \text{Since this integral represents the area of a circle of radius } a$$

■



a. One-sided (non-orientable) surface: a Möbius strip.



b. Two-sided (orientable) surface: a football. This is also a closed surface.

Figure 13.35 Surfaces in space

FLUX INTEGRALS

An important application of surface integrals involves the flow of a fluid through a surface. To discuss such applications, we define a surface S to be **orientable** if S has a unit normal vector field **N** that varies continuously over S. Most common surfaces, such as spheres, cones, cylinders, ellipsoids, and paraboloids, are orientable, but it is not difficult to construct a fairly simple surface that is not. For example, the **Möbius strip**, formed by twisting a long rectangular strip before joining the ends, is not orientable (see Figure 13.35a). If S is an orientable surface, there will be two possibilities for **N**, which can be thought of as **outward** (pointing toward the exterior of S) or **inward** (pointing toward the interior), as illustrated in Figure 13.35b.

Consider a fluid of density δ flowing steadily through a surface S with the continuous unit normal field **N**. If **V** is the velocity of the fluid flow, then the vector field $\mathbf{F} = \delta\mathbf{V}$ is the **flux density**, which measures the volume of fluid crossing the surface per unit area in unit time. Think of water flowing through a net, as shown in Figure 13.36.

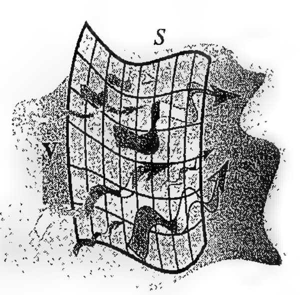


Figure 13.36 Water flowing through a net

Can leave it

If ΔS is the area of a small patch of the surface S, then the amount of fluid crossing this patch in unit time in the direction of **N** may be estimated by the volume of the cylinder of height $\mathbf{F} \cdot \mathbf{N}$, and base area ΔS (see Figure 13.37); that is,

$$\Delta V \approx (\mathbf{F} \cdot \mathbf{N})\Delta S$$

Thus, we can measure the total volume crossing the surface in unit time by computing the surface integral of $\mathbf{F} \cdot \mathbf{N}$, which is called a *flux integral*.

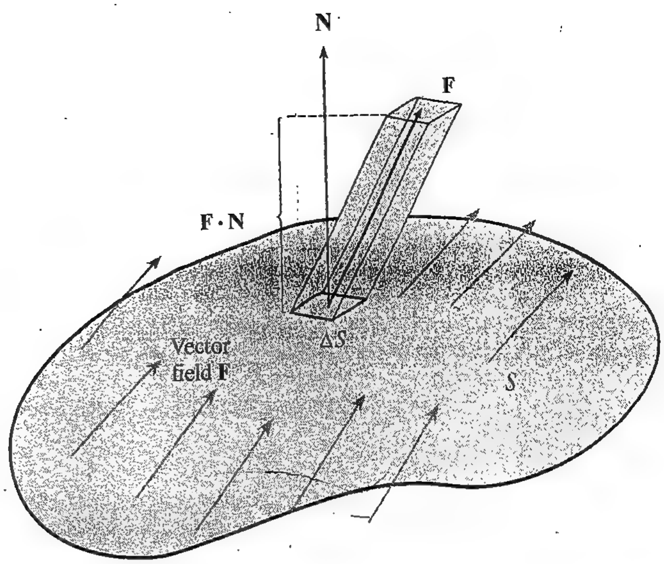


Figure 13.37 The cylinder of fluid crossing the patch of area ΔS in the direction of $\mathbf{N}$ has height $\mathbf{F} \cdot \mathbf{N}$ and volume $(\mathbf{F} \cdot \mathbf{N})\Delta S$

Flux Integral

Flux gives volume of fluid flowing across the surface in unit time

> Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on the surface S, which is oriented by the unit normal field $\mathbf{N}$. Then the **flux** of $\mathbf{F}$ across S is given by the surface integral
>
> $$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{N}\, dS$$

Suppose we have a surface S described by the equation $z = f(x, y)$ that projects onto a region D in the xy-plane. Then we have $G(x, y, z) = z - f(x, y)$ and an upward unit normal (the one with positive k-component) to the surface is given by

$$\mathbf{N} = \frac{\nabla G}{\|\nabla G\|} \quad \text{where } \nabla G = \langle -f_x, -f_y, 1\rangle$$

Therefore,

$$\mathbf{F} \cdot \mathbf{N}\, dS = \mathbf{F} \cdot \left(\frac{\nabla G}{\|\nabla G\|}\right) \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

and since $\|\nabla G\| = \sqrt{(-f_x)^2 + (-f_y)^2 + 1}$, the flux integral of $\mathbf{F}$ over S can be written as

$$\boxed{\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N}\, dS &= \iint_D \mathbf{F}(x, y, f(x, y)) \cdot \nabla G\, dA \\ &= \iint_D \mathbf{F}(x, y, f(x, y)) \cdot \langle -f_x, -f_y, 1\rangle\, dA \end{aligned}}$$

Similarly, a downward unit normal (negative k-component) to the surface is given by

$$\mathbf{N} = \frac{-\nabla G}{\|\nabla G\|}$$

and the flux integral has the form

$$\boxed{\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}(x, y, f(x, y))\cdot\langle f_x, f_y, -1\rangle\,dA}$$

Note that it is always necessary to specify the orientation of **N** (upward or downward) when describing a flux integral.

EXAMPLE 3 Evaluating a flux integral

Compute the flux integral $\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$, where $\mathbf{F} = xy\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k}$ and S is the triangular surface cut off from the plane $x + y + z = 1$ by the coordinate planes. Assume **N** is the upward unit normal.

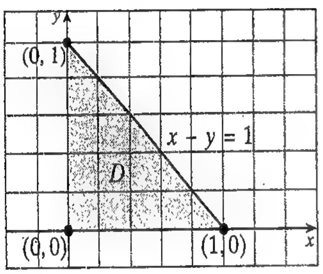


Figure 13.38 Projection D of $x + y + z = 1$ on the xy-plane

Solution

Let $f(x, y) = z = 1 - x - y$. Then $f_x = -1, f_y = -1$, and the flux integral is

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}(x, y, f(x, y))\cdot\langle -f_x, -f_y, 1\rangle\,dA$$

where D is the projection of S on the xy-plane. Setting $z = 0$, we find that D is the triangular region bounded by the lines $x + y = 1$, $x = 0$, and $y = 0$, as shown in Figure 13.38. Thus, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{N}\,dS &= \iint_D \langle xy, 1 - x - y, x + y\rangle\cdot\langle -(-1), -(-1), 1\rangle\,dA\\ &= \int_0^1\int_0^{1-x}[xy + (1 - x - y) + x + y]\,dy\,dx\\ &= \int_0^1\int_0^{1-x}[xy + 1]\,dy\,dx\\ &= \int_0^1 \left[\tfrac{1}{2}xy^2 + y\right]\Big|_0^{1-x}\,dx\\ &= \int_0^1 \tfrac{1}{2}(x^3 - 2x^2 - x + 2)\,dx\\ &= \tfrac{13}{24}\end{aligned}$$

■

EXAMPLE 4 Computing heat flow as a flux integral

Let R be the region that is bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the xy-plane. Experiments indicate that the velocity of heat flow is given by the vector field $\mathbf{H} = -K\nabla T$, where

$$T(x, y, z) = 2x + y - 3z^2$$

is the temperature at each point $P(x, y, z)$ in the region and K is a constant (the *heat conductivity*, which is obtained experimentally for each different substance). Find the total heat flow $\iint_R \mathbf{H}\cdot\mathbf{N}\,dS$ out of the region (that is, **N** is the outer unit normal, the one pointing away from the origin).

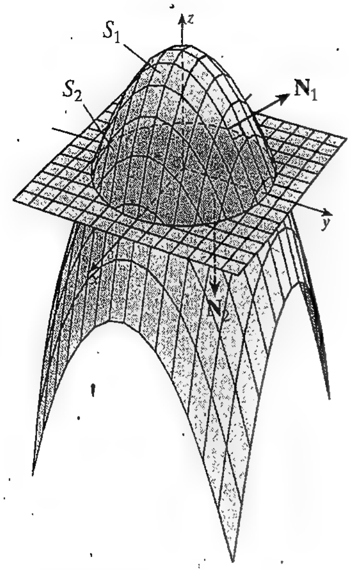


Figure 13.39 The surface S of the region bounded above by $z = 9 - x^2 - y^2$ and below by the xy-plane

Solution

The surface is shown in Figure 13.39. Note also that $\nabla T = \langle 2, 1, -6z \rangle$ and that the surface S is composed of a top surface S_1 and a bottom surface S_2. The top surface is S_1: $z = 9 - x^2 - y^2$ and the bottom surface is the disk S_2: $x^2 + y^2 \leq 9$, which is also the projection D of S_1 on the xy-plane.

For S_1: Let $G = z + x^2 + y^2 - 9$; then, $\nabla G = \langle 2x, 2y, 1 \rangle$. The top surface S_1 has upward pointing normal, and the flux integral over S_1 is

$$
\begin{aligned}
\iint_{S_1} \mathbf{H} \cdot \mathbf{N}_1 \, dS &= \iint_{S_1} -K \nabla T \cdot \nabla G \, dS \\
&= \iint_D -K \langle 2, 1, -6z \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA \\
&= \iint_D -K[4x + 2y - 6(9 - x^2 - y^2)] \, dA \qquad \text{Since } z = 9 - x^2 - y^2 \text{ on } S \\
&= -K \int_0^{2\pi} \int_0^3 [4r\cos\theta + 2r\sin\theta - 6(9 - r^2)] \, r \, dr \, d\theta \\
&= -K \int_0^{2\pi} \left[36\cos\theta + 18\sin\theta - \frac{243}{2} \right] d\theta \\
&= 243\pi K
\end{aligned}
$$

For S_2: The outer normal $\mathbf{N}_2 = -k = \langle 0, 0, -1 \rangle$ points downward and the flux integral is

$$
\begin{aligned}
\iint_{S_2} \mathbf{H} \cdot \mathbf{N}_2 \, dS &= \iint_D -K \langle 2, 1, -6z \rangle \cdot \langle 0, 0, -1 \rangle \, dA \\
&= -K \iint_D 6z \, dA \\
&= -K \iint_D 0 \, dA \qquad \text{Since } z = 0 \text{ on } S_2 \\
&= 0
\end{aligned}
$$

Thus, the total heat flow is

$$
\begin{aligned}
\iint_S \mathbf{H} \cdot \mathbf{N} \, dS &= \iint_{S_1} \mathbf{H} \cdot \mathbf{N}_1 \, dS + \iint_{S_2} \mathbf{H} \cdot \mathbf{N}_2 \, dS \\
&= 243\pi K + 0 = 243\pi K
\end{aligned}
$$

■

INTEGRALS OVER PARAMETRICALLY DEFINED SURFACES

If a surface S is defined parametrically by the vector function

$$\mathbf{R}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

over a region D in the uv-plane, the surface area of S is given by the integral

$$\iint_D \|\mathbf{R}_u \times \mathbf{R}_v\| \, du \, dv$$

A few param repres are on pg 158-189 CW copy

Similarly, if f is continuous on ~~D~~ S, the surface integral of f over ~~D~~ S is given by

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{R})\|\mathbf{R}_u \times \mathbf{R}_v\|\, du\, dv$$

EXAMPLE 5 Surface integral for a surface defined parametrically

Evaluate $\displaystyle\iint_S (x + y + z)\, dS$, where S is the surface defined parametrically by $\mathbf{R}(u, v) = (2u + v)\mathbf{i} + (u - 2v)\mathbf{j} + (u + 3v)\mathbf{k}$ for $0 \le u \le 1$, $0 \le v \le 2$.

Solution

$$\iint_S (x + y + z)\, dS = \iint_D f(\mathbf{R})\|\mathbf{R}_u \times \mathbf{R}_v\|\, du\, dv$$

First, we need to find the component parts for the surface integral on the right. Using $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we see that $x = 2u + v$, $y = u - 2v$, and $z = u + 3v$, and since $f(x, y, z) = x + y + z$, we have

$$\begin{aligned} f(\mathbf{R}) &= f(2u + v, u - 2v, u + 3v) \\ &= (2u + v) + (u - 2v) + (u + 3v) \\ &= 4u + 2v \end{aligned}$$

Next, we find that

$$\mathbf{R}_u = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{R}_v = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

so

$$\mathbf{R}_u \times \mathbf{R}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = (3 + 2)\mathbf{i} - (6 - 1)\mathbf{j} + (-4 - 1)\mathbf{k} = 5\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$$

Thus, $\|\mathbf{R}_u \times \mathbf{R}_v\| = \sqrt{5^2 + (-5)^2 + (-5)^2} = 5\sqrt{3}$. We now substitute these values into the surface integral formula:

$$\begin{aligned} \iint_S (x + y + z)\, dS &= \iint_D f(\mathbf{R})\|\mathbf{R}_u \times \mathbf{R}_v\|\, du\, dv \\ &= \int_0^2 \int_0^1 (4u + 2v)(5\sqrt{3})\, du\, dv = 5\sqrt{3} \int_0^2 \left[2u^2 + 2uv\right]\Big|_0^1 dv \\ &= 5\sqrt{3} \int_0^2 (2 + 2v)\, dv = 5\sqrt{3}\left[2v + v^2\right]\Big|_0^2 = 5\sqrt{3}(8) = 40\sqrt{3} \end{aligned}$$

■

For a flux integral over a surface S parameterized by $\mathbf{R}(u, v)$ that projects onto a region D in the uv-plane, we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N}\, dS &= \iint_D \mathbf{F} \cdot \mathbf{N}\|\mathbf{R}_u \times \mathbf{R}_v\|\, dA \\ &= \iint_D \mathbf{F} \cdot \left(\frac{\mathbf{R}_u \times \mathbf{R}_v}{\|\mathbf{R}_u \times \mathbf{R}_v\|}\right)\|\mathbf{R}_u \times \mathbf{R}_v\|\, dA \\ &= \iint_D \mathbf{F} \cdot (\mathbf{R}_u \times \mathbf{R}_v)\, dA \end{aligned}$$

The use of this formula is demonstrated in the next example.

EXAMPLE 6 Computing flux through a parameterized surface

Find the flux of the vector field $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + (y + z)\mathbf{k}$ through the parameterized surface

$$\mathbf{R}(u, v) = (uv)\mathbf{i} + (u - v)\mathbf{j} + (2u + v)\mathbf{k}$$

over the triangular region D in the uv-plane that is bounded by $u = 0$, $v = 0$, and $u + v = 1$.

Solution

First, note from $\mathbf{R}(u, v)$ that $x = uv$, $y = u - v$, and $z = 2u + v$, so we have

$$\begin{aligned}\mathbf{F}(u, v) &= (2u + v)\mathbf{i} + (uv)\mathbf{j} + [(u - v) + (2u + v)]\mathbf{k}\\ &= (2u + v)\mathbf{i} + (uv)\mathbf{j} + (3u)\mathbf{k}\end{aligned}$$

Moreover, since $\mathbf{R}_u = v\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{R}_v = u\mathbf{i} - \mathbf{j} + \mathbf{k}$, it follows that

$$\begin{aligned}\mathbf{R}_u \times \mathbf{R}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v & 1 & 2 \\ u & -1 & 1 \end{vmatrix}\\ &= 3\mathbf{i} - (v - 2u)\mathbf{j} + (-v - u)\mathbf{k}\end{aligned}$$

Thus, the flux is given by

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{N}\,dS &= \iint_D \mathbf{F}\cdot(\mathbf{R}_u \times \mathbf{R}_v)\,dA\\ &= \int_0^1\int_0^{1-u} \langle 2u + v, uv, 3u\rangle \cdot \langle 3, 2u - v, -(u + v)\rangle\,dv\,du\\ &= \int_0^1\int_0^{1-u} [(2u + v)(3) + (uv)(2u - v) + 3u(-u - v)]\,dv\,du\\ &= \int_0^1\int_0^{1-u} (2u^2v - 3u^2 - uv^2 - 3uv + 6u + 3v)\,dv\,du\\ &= \int_0^1 \frac{1}{6}(u - 1)(8u^3 - u^2 - 16u - 9)\,du\\ &= \frac{137}{120}\end{aligned}$$

■

RW③ (Pg90 – Pg.110)

~~Guidelines~~

13.5 PROBLEM SET

Q- 3, 5, 8, 10, 11, 14, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 36, 37.

A In Problems 1–4, evaluate $\iint_S xy\,dS$.

1. S: $z = 2 - y$, $0 \le x \le 2$, $0 \le y \le 2$
2. S: $z = 4 - x - y$, $0 \le x \le 4$, $0 \le y \le 4$
3. S: $z = 5$, $x^2 + y^2 \le 1$ 0
4. S: $z = 10$, $\dfrac{x^2}{4} + \dfrac{y^2}{1} \le 1$

In Problems 5–8, evaluate $\iint_S (x^2 + y^2)\,dS$.

5. S: $z = 4 - x - 2y$, $0 \le x \le 4$, $0 \le y \le 2$ $\frac{160\sqrt{6}}{3}$
6. S: $z = 4 - x$, $0 \le x \le 2$, $0 \le y \le 2$
7. S: $z = 4$, $x^2 + y^2 \le 1$
8. S: $z = xy$, $x^2 + y^2 \le 4$, $x \ge 0$, $y \ge 0$ $\pi(100\sqrt{5} + 4)$

Let S be the hemisphere $x^2 + y^2 + z^2 = 4$, with $z \ge 0$, in Problems 9–14. Evaluate each surface integral.

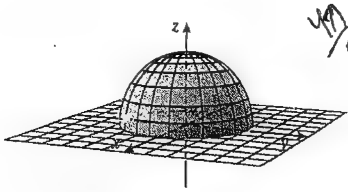
9. $\iint_S z\,dS$
10. $\iint_S z^2\,dS$ $\frac{32\pi}{3}$
11. $\iint_S (x - 2y)\,dS$ 0
12. $\iint_S (5 - 2x)\,dS$
13. $\iint_S (x^2 + y^2)z\,dS$
14. $\iint_S (x^2 + y^2)\,dS$ $\frac{64\pi}{3}$

In Problems 15–18, suppose S is the portion of the paraboloid $z = x^2 + y^2$ for which $z \le 4$. Evaluate the given surface integral.

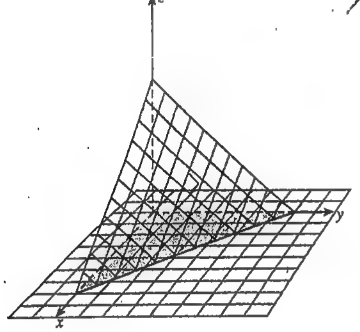
15. $\iint_S z\,dS$
16. $\iint_S (4 - z)\,dS$ $\frac{\pi}{60}(289\sqrt{17} - 41)$

17. $\iint_S \sqrt{1+4z}\, dS$ 18. $\iint_S \frac{dS}{\sqrt{1+4z}}$

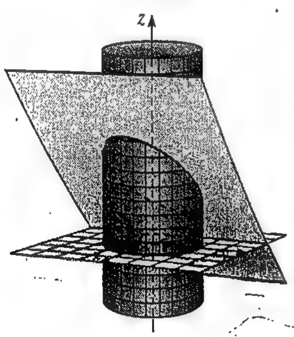
19. Evaluate $\iint_S (x^2+y^2)\, dS$, where S is the surface of the hemisphere $z = \sqrt{1-x^2-y^2}$.



20. Evaluate $\iint_S 2x\, dS$, where S is the portion of the plane $x+y+z=1$ with $x \geq 0, y \geq 0, z \geq 0$.



21. Evaluate $\iint_S (x^2+y^2+z^2)\, dS$, where S is the portion of the plane $z = x+1$ that lies inside the cylinder $x^2+y^2=1$.



Evaluate $\iint_S \mathbf{F}\cdot\mathbf{N}\, dS$ for the vector fields $\mathbf{F}$ *and surfaces S given in Problems 22–29. Assume* $\mathbf{N}$ *is the outward directed normal field.*

22. $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} + z\mathbf{k}$, and S is the triangular region bounded by the intersection of the plane $x+2y+z=1$ with the positive coordinate planes.

23. $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} - 3z\mathbf{k}$, and S is that part of the plane $15x - 12y + 3z = 6$ that lies above the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$.

24. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$, and S is the surface of the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0$, and $z=1$.

25. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, and S is the hemisphere $z = \sqrt{1-x^2-y^2}$.

26. $\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j}$, and S is the part of the hemisphere given by $x^2+y^2+z^2=5$, for $z \geq 1$.

27. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, and S is the portion of the plane $z = y+1$ that lies inside the cylinder $x^2+y^2=1$.

28. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, and S is the part of the cone $z = \sqrt{x^2+y^2}$ below the plane $z=2$.

29. $\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} - 2z\mathbf{k}$, and S is the part of the surface cut from the cylinder $z = 3 - x^2$ by $y=0, y=1$, and $z=0$.

B 30. Evaluate $\iint_S (x+y+z)\, dS$, where S is the surface determined by $\mathbf{R}(u,v) = u\mathbf{i} + u\mathbf{j} - v\mathbf{k}, 0 \leq u \leq 1, 1 \leq v \leq 2$.

31. Evaluate $\iint_S (x - y^2 + z)\, dS$, where S is the surface defined by $\mathbf{R}(u,v) = u^2\mathbf{i} + v\mathbf{j} + u\mathbf{k}, 0 \leq u \leq 1, 0 \leq v \leq 1$.

32. Evaluate $\iint_S (\tan^{-1} x + y - z^2)\, dS$, where S is the surface defined by $\mathbf{R}(u,v) = u\mathbf{i} + v^2\mathbf{j} - v\mathbf{k}, 0 \leq u \leq 1, 0 \leq v \leq 1$.

33. Evaluate $\iint_S (x^2 + y - z)\, dS$, where S is the surface defined by $\mathbf{R}(u,v) = u\mathbf{i} - u^2\mathbf{j} + v\mathbf{k}, 0 \leq u \leq 2, 0 \leq v \leq 1$.

34. Evaluate $\iint_S (x+y+z)\, dS$, where S is the surface of the cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

35. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{N}\, dS$, where $\mathbf{F} = x^2\mathbf{i} + z\mathbf{k}$ and S is the parametric surface

$$x = \sin u \cos v \qquad y = \sin u \sin v \qquad z = \cos u$$

for $0 \leq u \leq \pi, 0 \leq v \leq 2\pi$.

36. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{N}\, dS$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}$ and S is the parametric surface

$$x = u\cos v \qquad y = u \sin v \qquad z = u$$

for $0 \leq u \leq 2, 0 \leq v \leq 2\pi$.

In Problems 37–42 find the mass of the homogeneous lamina that has the shape of the given surface S.

37. S is the surface $z = 4 - x - 2y$, with $z \geq 0, x \geq 0, y \geq 0$; $\delta = x$.

38. S is the surface $z = 10 - 2x - y$, with $z \geq 0, x \geq 0, y \geq 0$; $\delta = y$.

39. S is the surface $z = x^2 + y^2$, with $z \leq 1$; $\delta = z$. *Hint:* Use cylindrical coordinates.

40. S is the surface $z = 1 - x^2 - y^2$, with $z \geq 0$; $\delta = x^2 + y^2 + z^2$. *Hint:* Use cylindrical coordinates.

41. S is the surface $r^2 + z^2 = 5$, with $z \geq 1$; $\delta = \theta^2$.

42. S is the triangular surface with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$; $\delta = x + y$.

43. A fluid with constant density ρ flows with velocity $\mathbf{V} = (xy)\mathbf{i} + (yz)\mathbf{j} + (xz)\mathbf{k}$. Find the upward rate of fluid flow through the paraboloid $z = 9 - x^2 - y^2$.

44. The temperature at each point (x, y, z) in a region D is $T = 3x^2 + 3z^2$. If the heat conductivity is $K = 5.8$, find the rate of heat flow outward across the cylinder $x^2 + z^2 = 4$ for $0 \leq y \leq 3$.

C 45. a.

b.

46. R

47. I

45. **a.** A lamina has the shape of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that lies within the cone $z = \sqrt{x^2 + y^2}$. Determine the mass of the lamina if $\delta(x, y, z) = x^2 y^2 z$.

b. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone whose vertex is at the origin and whose axis of symmetry coincides with the z-axis. Suppose the vertex angle of the cone is ϕ_0, with $0 \le \phi_0 < \pi/2$. Determine the mass of that portion of the sphere that is enclosed in the intersection of S and C. Assume $\delta(x, y, z) = 1$.

46. Recall the formula

$$I_z = \iint_S (x^2 + y^2)\delta(x, y, z)\, dS$$

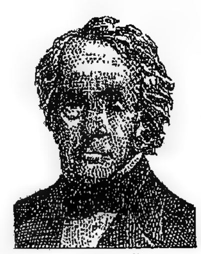
for the moment of inertia about the z-axis. Show that the moment of inertia of a conical shell about its axis is $\frac{1}{2}ma^2$, where m is the mass and a is the radius of the cone. Assume $\delta(x, y, z) = 1$.

47. Recall the formula

$$I_z = \iint_S (x^2 + y^2)\delta(x, y, z)\, dS$$

for the moment of inertia about the z-axis. Show that the moment of inertia of a spherical shell of uniform density about its diameter is $\frac{2}{3}ma^2$, where m is the mass and a is the radius. Assume $\delta(x, y, z) = 1$.

48. **HISTORICAL QUEST** August Möbius studied under Karl Gauss (1777–1855), as well as Gauss' own teacher Johann Pfaff (1765–1825). He was a professor at the University of Leipzig and is best known for his work in topology, especially for his conception of the Möbius strip (a two-dimensional surface with only one side). ■



AUGUST MÖBIUS
1790–1868

An example of a **Möbius strip** is given parametrically by:

$$x = \cos v + u \cos \frac{v}{2} \cos v$$
$$y = \sin v + u \cos \frac{v}{2} \sin v$$
$$z = u \sin \frac{v}{2}$$

where $-\frac{1}{2} \le u \le \frac{1}{2}$, $0 \le v \le 2\pi$. Construct a three-dimensional model of a Möbius strip and show that a Möbius strip is not orientable. If you have access to a CAS program, sketch the graph of this surface.

13.6 Stokes' Theorem

IN THIS SECTION Stokes' theorem, theoretical applications of Stokes' theorem, physical interpretation of Stokes' theorem

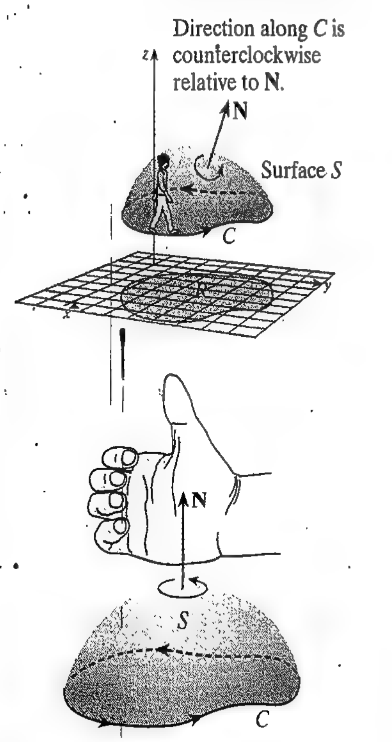


Figure 13.40 Compatible orientation: The surface S is on the left of someone walking in a counterclockwise direction around the boundary curve C

In Section 13.4, we observed that Green's theorem can be written

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_A (\text{curl}\, \mathbf{F} \cdot \mathbf{k})\, dA$$

where A is the plane region bounded by the Jordan curve C. Stokes' theorem is a generalization of this result to surfaces in space and their boundaries.

STOKES' THEOREM

Before stating Stokes' theorem, we need to explain what is meant by a *compatible orientation.* We say that the orientation of a closed path C on the surface S is **compatible with the orientation on** S if the positive direction on C is *counterclockwise* in relation to the outward normal vector **N** of the surface (see Figure 13.40). That is, if a person walks around C in a positive (counterclockwise) direction with her head pointing in the direction of **N**, then the surface will always be on her left.

THEOREM 13.8 **Stokes' theorem**

Let S be an oriented surface with unit normal vector field **N**, and assume that S is bounded by a piecewise smooth Jordan curve C whose orientation is compatible with that of S. If **F** is a vector field that is continuously differentiable on S, then

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\text{curl}\,\mathbf{F} \cdot \mathbf{N})\, dS$$

Proof A proof assuming **F**, *S*, and *C* are "well behaved" is given in Appendix B. ☐

EXAMPLE 1 Using Stokes' theorem to evaluate a line integral

Evaluate $\oint_C (\frac{1}{2}y^2\,dx + z\,dy + x\,dz)$, where C is the curve of intersection of the plane $x + z = 1$ and the ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented counterclockwise as viewed from above (see Figure 13.41).

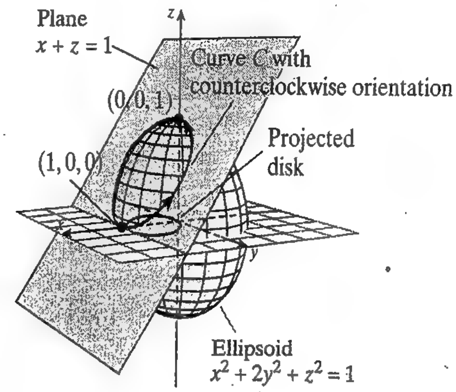


Figure 13.41 The surface bounded by the curve of intersection projects onto a disk in the xy-plane

Solution

If we set $\mathbf{F} = \frac{1}{2}y^2\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, the given line integral can be expressed as

$$\oint_C \mathbf{F} \cdot d\mathbf{R}$$

According to Stokes' theorem, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\text{curl}\,\mathbf{F} \cdot \mathbf{N})\, dS$$

There are many surfaces bounded by the curve C. We choose the surface S that is part of the plane $x + z = 1$. We find the required parts of the surface integral; namely, curl **F**, **N**, and dS.

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \frac{1}{2}y^2 & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} - y\mathbf{k}$$

The upward unit normal vector to the plane $x + z = 1$ is $\mathbf{N} = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$, so that

$$\begin{aligned} \text{curl}\,\mathbf{F} \cdot \mathbf{N} &= \langle -1, -1, -y \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle \\ &= -\frac{1}{\sqrt{2}} - \frac{y}{\sqrt{2}} \\ &= \frac{-1}{\sqrt{2}}(1 + y) \end{aligned}$$

and since $z = 1 - x$ on S, we have $z_x = -1$, $z_y = 0$, and

$$dS = \sqrt{z_x^2 + z_y^2 + 1}\,dA = \sqrt{(-1)^2 + (0)^2 + 1}\,dA = \sqrt{2}\,dA$$

Finally, to describe C we substitute $z = 1 - x$ into the equation for the ellipsoid:

$$\begin{aligned} x^2 + 2y^2 + z^2 &= 1 \\ x^2 + 2y^2 + (1-x)^2 &= 1 \\ x^2 - x + y^2 &= 0 \\ \left(x - \tfrac{1}{2}\right)^2 + y^2 &= \tfrac{1}{4} \end{aligned}$$

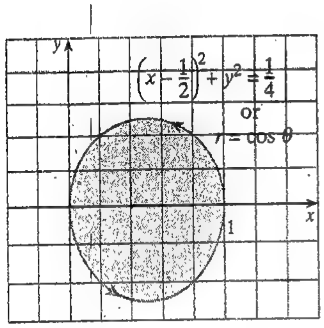


Figure 13.42 The projection D of the surface S on the xy-plane

Thus, the projection of C on the xy-plane is the disk D bounded by the circle $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$ shown in Figure 13.42. Note that this circle has the polar equation $r = \cos\theta$.

We now calculate the desired line integral using Stokes' theorem.

$$\begin{aligned} \oint_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right) &= \oint_C \mathbf{F}\cdot d\mathbf{R} \\ &= \iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{N})\,dS && \text{Stokes' theorem} \\ &= \iint_D \frac{-1}{\sqrt{2}}(1+y)(\sqrt{2}\,dA) \\ &= -\int_0^{\pi}\int_0^{\cos\theta}(1 + r\sin\theta)\,r\,dr\,d\theta && \text{Change to polar coordinates.} \\ &= -\int_0^{\pi}\left[\tfrac{1}{2}\cos^2\theta + \tfrac{1}{3}\cos^3\theta\sin\theta\right]d\theta = -\tfrac{\pi}{4} \end{aligned}$$

■

EXAMPLE 2 Verifying Stokes' theorem for a particular surface

Let S be the portion of the plane $x + y + z = 1$ that lies in the first octant, and let C be the boundary of S, traversed counterclockwise as viewed from above. Verify Stokes' theorem for the surface S and the vector field

$$\mathbf{F} = -\tfrac{3}{2}y^2\mathbf{i} - 2xy\mathbf{j} + yz\mathbf{k}$$

Solution

The surface S is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, whose boundary C is traversed in that order, as shown in Figure 13.43. We will show that the line integral $\oint_C \mathbf{F}\cdot d\mathbf{R}$ and the surface integral $\iint_S$ (curl $\mathbf{F}\cdot\mathbf{N}$) ds have the same value.

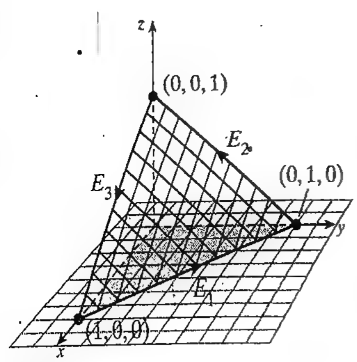


Figure 13.43 The curve C bounds the triangular surface S

I. *Evaluation of* $\oint_C \mathbf{F}\cdot d\mathbf{R}$

The three edges of the boundary curve C are expressed as

$$\begin{aligned} E_1&: \quad x + y = 1,\ z = 0 \\ E_2&: \quad y + z = 1,\ x = 0 \\ E_3&: \quad x + z = 1,\ y = 0 \end{aligned}$$

We traverse C in a counterclockwise direction (see Figure 13.43).

Edge E_1: Parameterize with $x = 1 - t$, $y = t$, $z = 0$, for $0 \leq t \leq 1$, so $\mathbf{R}(t) = (1 - t)\mathbf{i} + t\mathbf{j}$ and $d\mathbf{R} = -dt\mathbf{i} + dt\mathbf{j}$. Finally, in terms of the parameter t, we have $\mathbf{F}(t) = -\frac{3}{2}t^2\mathbf{i} - 2t(1-t)\mathbf{j}$.

$$\int_{E_1} \mathbf{F} \cdot d\mathbf{R} = \int_0^1 \left(\tfrac{3}{2}t^2 - 2t + 2t^2\right) dt$$
$$= \int_0^1 \left(\tfrac{7}{2}t^2 - 2t\right) dt = \left(\tfrac{7}{6}t^3 - t^2\right)\Big|_0^1 = \tfrac{1}{6}$$

Edge E_2: Parameterize with $x = 0$, $y = 1 - s$, $z = s$, for $0 \leq s \leq 1$, so $\mathbf{R}(s) = (1 - s)\mathbf{j} + s\mathbf{k}$ and $d\mathbf{R} = -ds\mathbf{j} + ds\mathbf{k}$. In terms of the parameter s, we have $\mathbf{F}(s) = -\frac{3}{2}(1-s)^2\mathbf{i} + (1-s)s\mathbf{k}$.

$$\int_{E_2} \mathbf{F} \cdot d\mathbf{R} = \int_0^1 (1-s)s\, ds = \left(\frac{s^2}{2} - \frac{s^3}{3}\right)\Bigg|_0^1 = \frac{1}{6}$$

Edge E_3: Parameterize with $x = r$, $y = 0$, $z = 1 - r$, for $0 \leq r \leq 1$, so $\mathbf{R}(r) = r\mathbf{i} + (1 - r)\mathbf{k}$ and $d\mathbf{R} = dr\mathbf{i} - dr\mathbf{k}$. In terms of the parameter r, we have $\mathbf{F}(r) = \mathbf{0}$.

$$\int_{E_3} \mathbf{F} \cdot d\mathbf{R} = 0$$

Combining these results, we find

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_{E_1} \mathbf{F} \cdot d\mathbf{R} + \int_{E_2} \mathbf{F} \cdot d\mathbf{R} + \int_{E_3} \mathbf{F} \cdot d\mathbf{R} = \frac{1}{6} + \frac{1}{6} + 0 = \frac{1}{3}$$

II. *Evaluation* of $\iint_S (\text{curl}\,\mathbf{F} \cdot \mathbf{N})\, dS$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\frac{3}{2}y^2 & -2xy & yz \end{vmatrix} = z\mathbf{i} + y\mathbf{k}$$

The triangular region S on the surface of the plane $x + y + z = 1$ has outward unit normal vector

$$\mathbf{N} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Because $z = 1 - x - y$, we find

$$\text{curl}\,\mathbf{F} \cdot \mathbf{N} = (z\mathbf{i} + y\mathbf{k}) \cdot \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$= \frac{1}{\sqrt{3}}(1 - x - y + y) = \frac{1}{\sqrt{3}}(1 - x)$$

and since $z_x = -1$, $z_y = -1$, it follows that

$$dS = \sqrt{z_x^2 + z_y^2 + 1}\, dA = \sqrt{(-1)^2 + (-1)^2 + 1}\, dA = \sqrt{3}\, dA$$

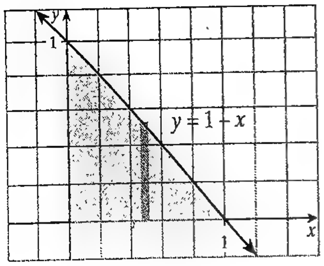


Figure 13.44 The projected region in the xy-plane is a triangle bounded by $y = 1 - x$ and the positive coordinate axes

Finally, the surface S projects onto the triangular region D in the xy-plane that is bounded by the lines $x = 0$, $y = 0$, and $x + y = 1$ (see Figure 13.44). Thus, we have

$$\iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{N})\,dS = \iint_D \frac{1}{\sqrt{3}}(1-x)\sqrt{3}\,dA = \int_0^1\int_0^{1-x}(1-x)\,dy\,dx$$

$$= \int_0^1 (1-x)[(1-x)-0]\,dx = -\frac{(1-x)^3}{3}\bigg|_0^1 = \frac{1}{3}$$

We see that for this example,

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \frac{1}{3} = \iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{N})\,dS$$

as claimed by Stokes' theorem. ■

Sometimes it is possible to exchange a particularly difficult surface integration over one surface for a less difficult integration over another surface with the same boundary curve. Suppose two surfaces S_1 and S_2 are bounded by the same curve C and induce the same orientation on C. Stokes' theorem tells us that

$$\iint_{S_1} (\operatorname{curl}\mathbf{F}\cdot\mathbf{N}_1)\,dS_1 = \oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_{S_2} (\operatorname{curl}\mathbf{F}\cdot\mathbf{N}_2)\,dS_2$$

for any function $\mathbf{F}$ whose components have continuous partial derivatives on both S_1 and S_2.

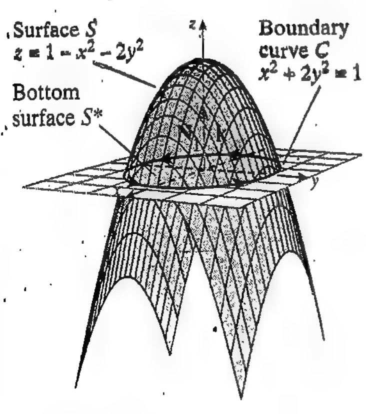


Figure 13.45 Surface $z = 1 - x^2 - 2y^2$ and boundary curve C

EXAMPLE 3 Using Stokes' theorem to evaluate a surface integral

Evaluate $\iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{N})\,dS$, where $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + ze^{xy}\mathbf{k}$ and S is that part of the surface $z = 1 - x^2 - 2y^2$ with $z \geq 0$.

Solution

By setting $z = 0$ in the equation of the surface, we find that the boundary curve C for S is the ellipse $x^2 + 2y^2 = 1$, and the upward unit normal vector $\mathbf{N}$ on S induces a counterclockwise orientation on C, as shown in Figure 13.45. Let S^* be the elliptical disk defined by $x^2 + 2y^2 \leq 1$, $z = 0$. We see that S and S^* have the same boundary and the same orientation. Since the upward unit normal to S^* is $\mathbf{k}$, we have

$$\iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{N})\,dS = \iint_{S^*} (\operatorname{curl}\mathbf{F}\cdot\mathbf{k})\,dS^*$$

and we use this second integral to calculate the required integral. We obtain

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y^2 & ze^{xy} \end{vmatrix} = (zxe^{xy})\mathbf{i} - (zye^{xy})\mathbf{j}$$

Since $z = 0$ on S^*, we have $\mathbf{F} \cdot \mathbf{k} = 0$ on S^* and

$$\iint_S (\text{curl}\,\mathbf{F} \cdot \mathbf{N})\, dS = \iint_{S^*} (\text{curl}\,\mathbf{F} \cdot \mathbf{k})\, dS^* = \iint_{S^*} 0\, dS^* = 0 \qquad \blacksquare$$

THEORETICAL APPLICATIONS OF STOKES' THEOREM

In physics and other applied areas, Stokes' theorem is often used as a device for establishing general properties. For instance, we now use it to prove the curl criterion for a conservative vector field in $\mathbb{R}^3$ stated without proof in Section 13.3 (Theorem 13.4), and whose proof was promised at this time.

The curl criterion for a conservative vector field in $\mathbb{R}^3$ Suppose the vector field $\mathbf{F}$ and curl $\mathbf{F}$ are both continuous in the simply connected region D of $\mathbb{R}^3$. Then $\mathbf{F}$ is conservative in D if and only if curl $\mathbf{F} = \mathbf{0}$ in D.

Proof of Theorem 13.4

If $\mathbf{F}$ is conservative, let f be a scalar potential function, so that $\nabla f = \mathbf{F}$. Then curl $\mathbf{F} = \nabla \times \mathbf{F} = \nabla \times (\nabla f) = \mathbf{0}$. (See Problem 54, Section 13.1 for a proof of this property.)

Conversely, if curl $\mathbf{F} = \mathbf{0}$, we will show that $\mathbf{F}$ is conservative by proving that it is independent of path. Let P_1 and P_2 be any two points in the simply connected region D and let C_1 and C_2 be two nonintersecting paths in D from P_1 to P_2. Let C be the Jordan curve from P_1 back to P_1 formed by C_1 followed by $-C_2$. Since D is simply connected (no holes), there is a piecewise smooth surface S whose boundary is C. Then, by Stokes' theorem

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{R} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{R} - \int_{C_2} \mathbf{F} \cdot d\mathbf{R} \\ &= \iint_S \text{curl}\,\mathbf{F} \cdot \mathbf{N}\, dS \\ &= 0\end{aligned}$$

so that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{R} = \int_{C_2} \mathbf{F} \cdot d\mathbf{R}$$

It follows that the line integral $\oint_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path, so $\mathbf{F}$ must be conservative. ❑

Maxwell's equations describing the behavior of electromagnetic phenomena were discussed briefly at the end of Section 13.1. The following example shows how Stokes' theorem can be used to establish one of Maxwell's equations, the *current density equation.*

EXAMPLE 4 Maxwell's current density equation

In physics, it is shown that if I is the current crossing any surface S bounded by the closed curve C, then

$$\oint_C \mathbf{H} \cdot d\mathbf{R} = I \quad \text{and} \quad \iint_S \mathbf{J} \cdot \mathbf{N}\, dS = I$$

where $\mathbf{H}$ is the magnetic intensity, and $\mathbf{J}$ is the electric current density. Use this information to derive Maxwell's current density equation curl $\mathbf{H} = \mathbf{J}$.

Solution

Equating the two equations of current, we obtain

$$\oint_C \mathbf{H} \cdot d\mathbf{R} = I = \iint_S \mathbf{J} \cdot \mathbf{N}\, dS$$

By Stokes' theorem,

$$\oint_C \mathbf{H} \cdot d\mathbf{R} = \iint_S (\operatorname{curl} \mathbf{H} \cdot \mathbf{N})\, dS$$

Equating the two surface integrals that equal $\oint_C \mathbf{H} \cdot d\mathbf{R}$, we have

$$\iint_S \mathbf{J} \cdot \mathbf{N}\, dS = \iint_S (\operatorname{curl} \mathbf{H} \cdot \mathbf{N})\, dS$$

or, equivalently,

$$\iint_S (\mathbf{J} - \operatorname{curl} \mathbf{H}) \cdot \mathbf{N}\, dS = 0$$

Because this equation holds for *any* surface S bounded by C, it can be shown that

$$\mathbf{J} - \operatorname{curl} \mathbf{H} = \mathbf{0}$$
$$\mathbf{J} = \operatorname{curl} \mathbf{H}$$

■

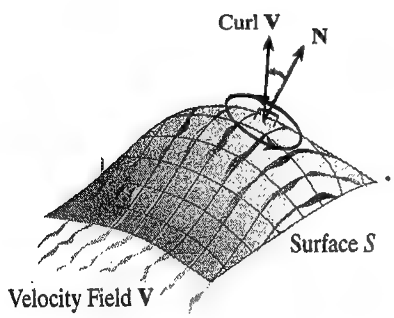
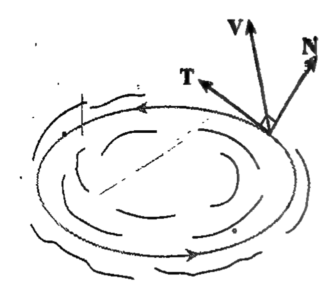


Figure 13.46 The tendency of a fluid to swirl across the surface S is measured by curl $\mathbf{V} \cdot \mathbf{N}$

PHYSICAL INTERPRETATION OF STOKES' THEOREM

If V is the velocity field of a fluid flow, then curl V measures the tendency of the fluid to rotate, or swirl (see Figure 13.46). If the fluid flows across the surface S, the rotational tendency usually will vary from point to point on the surface, and the surface integral $\iint_S (\operatorname{curl} \mathbf{V} \cdot \mathbf{N})\, dS$ provides a measure of the *cumulative* rotational tendency over the entire surface.

Stokes' theorem tells us that this cumulative measure of rotational tendency equals the line integral $\oint_C \mathbf{V} \cdot d\mathbf{R}$. To interpret this line integral, recall that it can be written $\oint_C \mathbf{V} \cdot \mathbf{T}\, ds$ in terms of the arc length parameter s and the unit tangent $\mathbf{T}$ to the curve. Since the line integral sums the *tangential component* of the velocity field $\mathbf{V}$, it measures the rate of flow of fluid mass around C and for this reason is called the **circulation of V** around C. If curl $\mathbf{V} = \mathbf{0}$, the circulation is zero and $\mathbf{V}$ is said to be **irrotational.** To summarize,



$$\underbrace{\iint_S (\operatorname{curl} \mathbf{V} \cdot \mathbf{N})\, dS}_{\substack{\text{The cumulative tendency} \\ \text{of a fluid to swirl across} \\ \text{the surface } S}} = \underbrace{\oint_C \mathbf{V} \cdot \mathbf{T}\, ds}_{\substack{\text{The circulation of} \\ \text{a fluid around the} \\ \text{boundary curve } C}}$$

Guidelines (@ 1, 2, 3, 4, 13, 16, 17, 18, 19, 20, 27, 31)

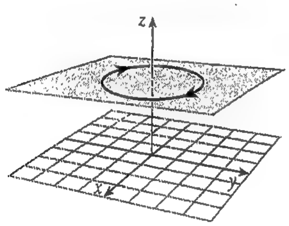
13.6 PROBLEM SET

A *Verify Stokes' theorem for the vector functions and surfaces given in Problems 1–5.*

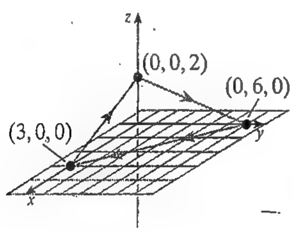
1. $\mathbf{F} = z\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$; S is the upper hemisphere $z = \sqrt{9 - x^2 - y^2}$. 18π
2. $\mathbf{F} = (y + z)\mathbf{i} + x\mathbf{j} + (z - x)\mathbf{k}$; S is the triangular region of the plane $x + 2y + z = 3$ in the first octant. 0
3. $\mathbf{F} = (x + 2z)\mathbf{i} + (y - x)\mathbf{j} + (z - y)\mathbf{k}$; S is the triangular region with vertices $(3, 0, 0)$ $(0, \frac{3}{2}, 0)$, $(0, 0, 3)$.
4. $\mathbf{F} = 2xy\mathbf{i} + z^2\mathbf{k}$; S is the portion of the paraboloid $y = x^2 + z^2$, with $y \le 4$. 0
5. $\mathbf{F} = 2y\mathbf{i} - 6z\mathbf{j} + 3x\mathbf{k}$; S is the portion of the paraboloid $z = 4 - x^2 - y^2$ and above the xy-plane.

Use Stokes' theorem to evaluate the line integrals given in Problems 6–13.

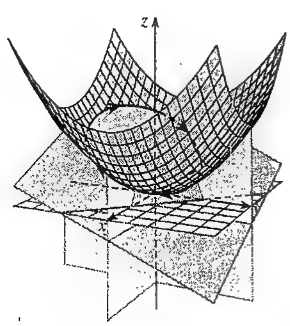
6. $\oint_C (x^3y^2\,dx + dy + z^2\,dz)$, where C is the circle $x^2 + y^2 = 1$ in the plane $z = 1$, counterclockwise when viewed from the origin



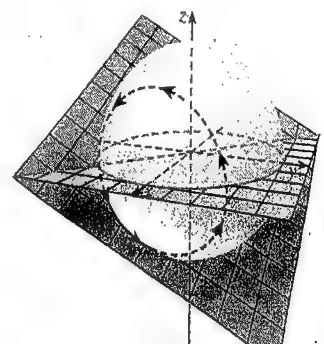
7. $\oint_C (z\,dx + x\,dy + y\,dz)$, where C is the triangle with vertices $(3, 0, 0)$, $(0, 0, 2)$, and $(0, 6, 0)$, traversed in the given order



8. $\oint_C (y\,dx - 2x\,dy + z\,dz)$, where C is the intersection of the surface $z = x^2 + y^2$ and the plane $x + y + z = 1$ considered counterclockwise when viewed from the origin



9. $\oint_C [2xy^2z\,dx + 2x^2yz\,dy + (x^2y^2 - 2z)\,dz]$, where C is the curve given by $x = \cos t$, $y = \sin t$, $z = \sin t$, $0 \le t \le 2\pi$, traversed in the direction of increasing t
10. $\oint_C (y\,dx + z\,dy + y\,dz)$, where C is the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $x + y + z = 0$, traversed counterclockwise when viewed from above



11. $\oint_C (y\,dx + z\,dy + x\,dz)$, where C is the intersection of the plane $x + y = 2$ and the surface $x^2 + y^2 + z^2 = 2(x + y)$, traversed counterclockwise as viewed from the origin
12. $\oint_C [(z + \cos x)\,dx + (x + y^2)\,dy + (y + e^z)\,dz]$, where C is the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the cone $z = \sqrt{x^2 + y^2}$, traversed counterclockwise as viewed from above
13. $\oint_C (3y\,dx + 2z\,dy - 5x\,dz)$, where C is the intersection of the xy-plane and the hemisphere $z = \sqrt{1 - x^2 - y^2}$, transversed counterclockwise as viewed from above 3π
14. $\oint_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = \langle x - z, y - x, z - y \rangle$ and C is the boundary of the triangular region with vertices $(12, 0, 0)$, $(0, 3, 0)$, $(0, 0, 12)$ traversed counterclockwise as viewed from above
15. $\oint_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = \langle y^2 + z^2, x^2 + y^2, x^2 + y^2 \rangle$ and C is the triangle $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ traversed in that order

In Problems 16–23, use Stokes' theorem to evaluate

$$\iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{N})\,dS$$

for the prescribed vector fields and surfaces. In each case, use the upward unit normal for S.

16. $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + xyz\mathbf{k}$, and S is the part of the paraboloid $z = 4 - x^2 - y^2$ with $z \ge 0$. Use the upward unit normal vector. 0
17. $\mathbf{F} = xy\mathbf{i} - z\mathbf{j}$, and S is the surface of the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ except for the face where $z = 0$. 1/2

18. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, and S is the part of the plane $x + y + z = 1$ that lies in the first octant.

19. $\mathbf{F} = xy\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k}$, and S is the part of the plane $z = y$ that is inside the paraboloid $z = x^2 + y^2$.

20. $\mathbf{F} = xz\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$, and S is the part of the plane $x + y + z = 3$ inside the cylinder $9x^2 + y^2 = 9$.

21. $\mathbf{F} = 4y\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$, and S is the hemisphere $z = \sqrt{4 - x^2 - y^2}$.

22. $\mathbf{F} = \langle x \tan^{-1} e^{-x}, y \ln(1 + y^{3/2}), ze^{-1/z} \rangle$, and S is the part of the sphere $x^2 + y^2 + z^2 = 9$ that lies inside the cone $z = \sqrt{2x^2 + 2y^2}$.

23. $\mathbf{F} = \langle 6x^2e^{yz}, 2x^3ze^{yz}, 2x^3ye^{yz} \rangle$, and S is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies above the inner loop of the limaçon $r = 1 + 2\cos\theta$.

B *In Problems 24–26, use Stokes' theorem to evaluate the line integral*

$$\oint_C [(1 + y)z\,dx + (1 + z)x\,dy + (1 + x)y\,dz]$$

for the given closed path C.

24. C is the elliptic path $x = 2\cos\theta, y = \sin\theta, z = 1$ for $0 \le \theta \le 2\pi$.

25. C is the boundary of the triangle with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1).

26. C is any closed path in the plane $2x - 3y + z = 1$.

In Problems 27–30, the vector field **V** *represents the velocity of a fluid flow. In each case, find the circulation*

$$\oint_C \mathbf{V} \cdot d\mathbf{R}$$

around the boundary C, assuming a counterclockwise orientation as viewed from above.

27. $\mathbf{V} = x\mathbf{i} + (z - x)\mathbf{j} + y\mathbf{k}$, and C is the intersection of the cylinder $x^2 + y^2 = y$ and the hemisphere $z = \sqrt{1 - x^2 - y^2}$.

28. $\mathbf{V} = y\mathbf{i} + \ln(x^2 + y^2)\mathbf{j} + (x + y)\mathbf{k}$, and C is the triangle with vertices (0, 0, 0), (1, 0, 0), and (0, 1, 0).

29. $\mathbf{V} = (e^{x^2} + z)\mathbf{i} + (x + \sin y^3)\mathbf{j} + [y + \ln(\tan^{-1} z)]\mathbf{k}$, and C is the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

30. $\mathbf{V} = y^2\mathbf{i} + \tan^{-1} z\mathbf{j} + (x^2 + 1)\mathbf{k}$, and C is the intersection of the plane $z = y$ and the cylinder $x^2 + y^2 = 2x$.

31. Let $\mathbf{F} = y^2\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$, and suppose S is the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \ge 0$. Use Stokes' theorem to express

$$\iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{N})\,dS$$

as a line integral, and then evaluate the surface integral by evaluating this line integral.

32. Let $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$, and suppose S is a smooth surface in $\mathbb{R}^3$ whose boundary is given by $x = 2\cos\theta$, $y = 3\sin\theta$, $z = \sin\theta$, $0 \le \theta \le 2\pi$. Use Stokes' theorem to evaluate

$$\iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{N})\,dS$$

33. **Counterexample Problem** Let C_1 and C_2 be positively oriented Jordan curves in the plane $5x + 3y + 2z = 4$ that contain the same area, and let $\mathbf{F} = \langle 4z, -3x, 2y \rangle$. Either prove that

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{R} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{R}$$

or find a counterexample.

34. **HISTORICAL QUEST** George Stokes was an English mathematical physicist who made important contributions to fluid mechanics, including the Navier-Stokes equations, which are important for modeling fluid flow. Most of his research was done before 1850, after which he held the Lucasian chair of mathematics at Cambridge for the better part of a half-century. William Thomson (Lord Kelvin) knew the result now known as Stokes' theorem in 1850 and sent it to Stokes as a challenge. Stokes proved the theorem, and then included it as an exam question in 1854. One of the students taking this particular examination was James Clerk Maxwell (Historical Quest #32, Section 13.7), who derived the famous electromagnetic wave equations ten years later. ■

GEORGE STOKES
1819–1903

For this Quest, write a history of the Lucasian chair, which was deeded in 1663 as a gift of Henry Lucas. The first and second appointees were Isaac Barrow (Historical Quest #5, page 28) and Isaac Newton (Historical Quest #1, page 28); and the chair is currently held by Stephen Hawking.

C 35. Let S be the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$, and let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on S. Use Stokes' theorem to show that

$$\iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{N})\,dS = 0$$

Does it matter that S is an ellipsoid? State and prove a more general result based on what you have discovered in the first part of this problem.

36. **Faraday's law** of electromagnetism says that if $\mathbf{E}$ is the electric intensity vector in a system, then

$$\oint_C \mathbf{E} \cdot d\mathbf{R} = -\frac{\partial \phi}{\partial t}$$

around any closed curve C, where t is time and ϕ is the total magnetic flux directed outward through any surface S bounded by C. Given that

$$\phi = \iint_S \mathbf{B} \cdot \mathbf{N}\,dS$$

where $\mathbf{B}$ is the magnetic flux density, show that $\operatorname{curl} \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$.

Hint: It can be shown that $\dfrac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathbf{N}\,dS = \iint_S \dfrac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{N}\,dS$.

37. The current I flowing across a surface S bounded by the closed curve C is given by

$$I = \iint_S \mathbf{J} \cdot \mathbf{N}\, dS$$

where $\mathbf{J}$ is the current density. Given that $\mu\mathbf{J} = \operatorname{curl} \mathbf{B}$, where $\mathbf{B}$ is magnetic flux density and μ is a constant, show that

$$\oint_C \mathbf{B} \cdot d\mathbf{R} = \mu I$$

Suppose f and g are functions of x, y, and z with continuous first- and second-order partial derivatives and C is a closed curve bounding the surface S. Use Stokes' theorem to verify the formulas given in Problems 38–39.

38. $\displaystyle\oint_C (f\nabla g) \cdot d\mathbf{R} = \iint_S (\nabla f \times \nabla g) \cdot \mathbf{N}\, dS$

39. $\displaystyle\oint_C (f\nabla g + g\nabla f) \cdot d\mathbf{R} = 0$

13.7 The Divergence Theorem

IN THIS SECTION the divergence theorem, applications of the divergence theorem, physical interpretation of divergence

THE DIVERGENCE THEOREM

We used Green's theorem to show that $\displaystyle\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \operatorname{div} \mathbf{F}\, dA$, where D is a simply connected domain with the closed boundary curve C. The *divergence theorem* (also known as **Gauss' theorem**) is a generalization of this form of Green's theorem that relates an integral over a closed surface to a volume integral.

see fmb(8) pg 139

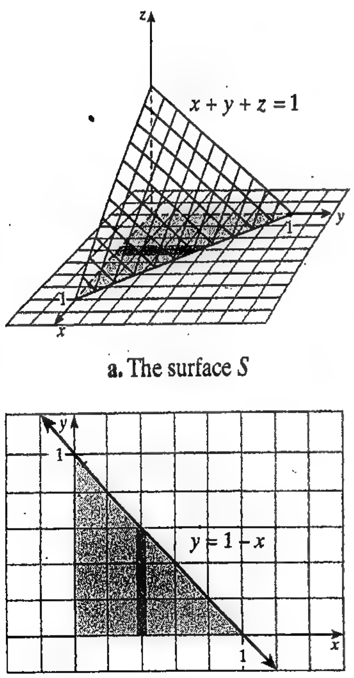
***THEOREM 13.9* The divergence theorem**

Let S be a smooth, orientable surface that encloses a solid region D in $\mathbb{R}^3$. If $\mathbf{F}$ is a continuous vector field whose components have continuous partial derivatives in an open set containing D, then

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV$$

where $\mathbf{N}$ is the outward unit normal field for the surface S.

Proof An important special case is proved in Appendix B. ❑



a. The surface S

b. Projected region in the xy-plane

Figure 13.47 A tetrahedron in $\mathbb{R}^3$

EXAMPLE 1 Evaluating a surface integral using the divergence theorem

Evaluate $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$, where $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + x^3y^3\mathbf{k}$ and S is the surface of the tetrahedron bounded by the plane $x + y + z = 1$ and the coordinate planes, with outward unit normal vector $\mathbf{N}$ (see Figure 13.47).

Solution

We will use the divergence theorem. Note that

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(x^3y^3) = 2x + x + 0 = 3x$$

The tetrahedron is the set

$$R: \quad \text{all } (x, y, z) \text{ such that } 0 \le z \le 1 - x - y \text{ whenever } 0 \le y \le 1 - x \text{ for } 0 \le x \le 1$$

This projects onto the triangular region in the xy-plane described by

$$D: \quad \text{all } (x, y) \text{ such that } 0 \leq y \leq 1 - x \text{ for } 0 \leq x \leq 1$$

(See Figure 13.47b.) Then, by applying the divergence theorem, we find that

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_R \operatorname{div} \mathbf{F}\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 3x\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} 3x(1 - x - y)\, dy\, dx = 3\int_0^1 \left[x(1-x)y - \tfrac{1}{2}xy^2\right]\Big|_0^{1-x} dx$$

$$= 3\int_0^1 \left[x(1-x)^2 - \tfrac{1}{2}x(1-x)^2\right] dx = \tfrac{1}{8}$$ ■

ok **EXAMPLE 2 Verifying the divergence theorem for a particular solid**

Let $\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j} + 5z\mathbf{k}$, and let S be the hemisphere $z = \sqrt{9 - x^2 - y^2}$ together with the disk $x^2 + y^2 \leq 9$ in the xy-plane. Verify the divergence theorem.

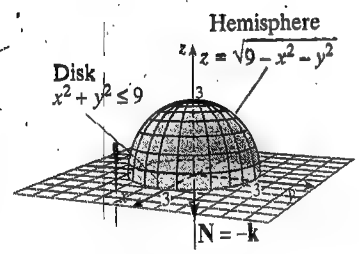


Figure 13.48 The surface of the hemisphere $z = \sqrt{9 - x^2 - y^2}$

Solution

The solid is shown in Figure 13.48. We will show that the surface integral $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$ and the triple integral $\iiint_R \operatorname{div} \mathbf{F}\, dV$ have the same value, where R is the solid bounded by S.

I. *Evaluation* of $\iiint_R \operatorname{div} \mathbf{F}\, dV$

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-3y) + \frac{\partial}{\partial z}(5z) = 2 - 3 + 5 = 4$$

Therefore, $\iiint_R \operatorname{div} \mathbf{F}\, dV = \iiint_R 4\, dV$, but $\iiint_R dV$ is just the volume of the hemisphere $z = \sqrt{9 - x^2 - y^2}$. A hemisphere of radius 3 has volume $\frac{1}{2}\left[\frac{4}{3}\pi(3)^3\right] = 18\pi$, so

$$\iiint_R \operatorname{div} \mathbf{F}\, dV = \iiint_R 4\, dV = 4V = 4(18\pi) = 72\pi$$

II. *Evaluation* of $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$

S consists of two parts: S_1, the disk on the bottom of the hemisphere, and S_2, the hemisphere. We will consider these separately and then use

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{N}_1\, dS_1 + \iint_{S_2} \mathbf{F} \cdot \mathbf{N}_2\, dS_2$$

The surface S_1: The disk $x^2 + y^2 \leq 9$ with $z = 0$ has outward (downward) unit normal $\mathbf{N} = -\mathbf{k}$, so

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{N}\,dS_1 = \iint_{S_1} \langle 2x, -3y, 5z\rangle \cdot \langle 0, 0, -1\rangle\, dS_1$$

$$= \iint_{S_1} (-5z)\, dS_1 = 0 \quad \textsf{Because } z = 0 \textsf{ on } S_1$$

The surface S_2: Since $z = \sqrt{9 - x^2 - y^2}$, we have

$$z_x = \frac{-x}{\sqrt{9 - x^2 - y^2}} \quad \text{and} \quad z_y = \frac{-y}{\sqrt{9 - x^2 - y^2}}$$

and the projection of S_2 onto the xy-plane is the disk D: $x^2 + y^2 \le 9$ or $r \le 3$ in polar form. (See Figure 13.49.) We find that

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{N}\,dS_2 = \iint_D \langle 2x, -3y, 5z\rangle \cdot \left\langle -\left(\frac{-x}{\sqrt{9-x^2-y^2}}\right), -\left(\frac{-y}{\sqrt{9-x^2-y^2}}\right), 1\right\rangle dA$$

$$= \iint_D \left[\frac{2x^2 - 3y^2}{\sqrt{9-x^2-y^2}} + 5z\right] dA$$

$$= \iint_D \left[\frac{2x^2 - 3y^2}{\sqrt{9-x^2-y^2}} + 5\sqrt{9-x^2-y^2}\right] dA \quad \textsf{Since } z = \sqrt{9-x^2-y^2} \textsf{ on } S$$

$$= \int_0^{2\pi}\int_0^3 \left[\frac{2r^2\cos^2\theta - 3r^2\sin^2\theta}{\sqrt{9-r^2}} + 5\sqrt{9-r^2}\right] r\,dr\,d\theta \quad \textsf{Changing to polar coordinates}$$

$$= \int_0^{2\pi} \left[81 - 90\sin^2\theta\right] d\theta$$

$$= 72\pi$$

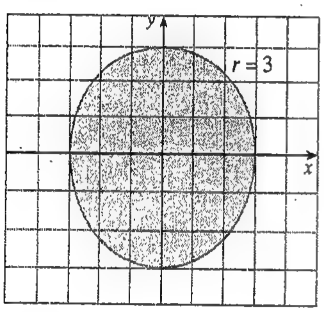


Figure 13.49 The hemisphere projects onto the disk $x^2 + y^2 \le 9$, or $r \le 3$ in polar form

Adding the surface integrals for S_1 and S_2, we obtain

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{N}\,dS_1 + \iint_{S_2} \mathbf{F}\cdot\mathbf{N}\,dS_2 = 0 + 72\pi = 72\pi$$

Thus, we have

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = 72\pi = \iiint_R \operatorname{div}\mathbf{F}\,dV$$

as required by the divergence theorem. ■

The divergence theorem applies only to closed surfaces. However, if we wish to evaluate $\iint_{S_1} \mathbf{F}\cdot\mathbf{N}\,dS$ where S_1 is *not* closed, we may be able to find a closed surface S that is the union of S_1 and some other surface S_2. Then, if the hypotheses of the divergence theorem are satisfied by $\mathbf{F}$ and S, we have

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{N}\,dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{N}\,dS = \iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_D \operatorname{div}\mathbf{F}\,dV$$

where D is the solid region bounded by S. Thus, if we can compute $\iiint_D \operatorname{div} \mathbf{F}\, dV$ and $\iint_{S_2} \mathbf{F} \cdot \mathbf{N}\, dS$, we can compute $\iint_{S_1} \mathbf{F} \cdot \mathbf{N}\, dS$ by the equation

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV - \iint_{S_2} \mathbf{F} \cdot \mathbf{N}\, dS$$

This equation can also be used as a device for trading the evaluation of a difficult surface integral for that of an easier volume integral. Here is an example of this procedure.

EXAMPLE 3 Evaluating a surface integral over an open surface

Evaluate $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$, where $\mathbf{F} = xy\mathbf{i} - z^2\mathbf{k}$ and S is the surface of the upper five faces of the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 < z \le 1$, as shown in Figure 13.50.

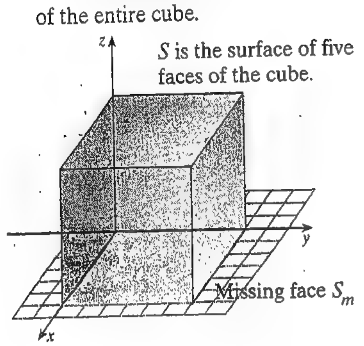


Figure 13.50 An open surface: a cube with a missing face

Solution

Note that the surface S is not closed, but we can close it by adding the missing face S_m, thus forming a closed surface S^* that satisfies the conditions of the divergence theorem. The strategy is to evaluate the surface integral of S^* and then subtract the surface integral over the added face S_m.

$$\iint_{S^*} \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_{\text{cube}} \operatorname{div} \mathbf{F}\, dV = \int_0^1 \int_0^1 \int_0^1 (y - 2z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 (y - 2z)\, dy\, dz = \int_0^1 \left(\tfrac{1}{2} - 2z\right) dz = -\tfrac{1}{2}$$

Also, because the outward unit normal vector to the added face S_m is $\mathbf{N} = -\mathbf{k}$ and $z = 0$ on this face, it follows that

$$\iint_{S_m} \mathbf{F} \cdot \mathbf{N}\, dS = \iint_{S_m} (xy\mathbf{i} - z^2\mathbf{k}) \cdot (-\mathbf{k})\, dS = \iint_{S_m} z^2\, dS = 0$$

Therefore,

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV - \iint_{S_m} \mathbf{F} \cdot \mathbf{N}\, dS = -\tfrac{1}{2} - 0 = -\tfrac{1}{2}$$ ■

APPLICATIONS OF THE DIVERGENCE THEOREM

Like Stokes' theorem, the divergence theorem is often used for theoretical purposes, especially as a tool for deriving general properties in mathematical physics. The following example deals with an important property of fluid dynamics.

EXAMPLE 4 Continuity equation of fluid dynamics

Suppose a fluid with density $\delta\,(x, y, z, t)$ flows in some region of space with velocity $\mathbf{F}(x, y, z, t)$ at the point (x, y, z) at time t. Assuming there are no sources or sinks, show that

$$\operatorname{div} \delta\mathbf{F} = -\frac{\partial \delta}{\partial t}$$

Solution

Recall from Section 13.1 that a point is called a *source* if div $\delta\mathbf{F} > 0$, a *sink* if div $\delta\mathbf{F} < 0$, and *incompressible* if div $\delta\mathbf{F} = 0$. Let S be a smooth surface in $\mathbb{R}^3$ that encloses a solid region D. In physics, it is shown that the amount of fluid flowing out of D across S in unit time is $\iint_S \delta\mathbf{F}\cdot\mathbf{N}\,dS$, which must equal the net decrease in density of the fluid; namely $\iiint_D \frac{\partial\delta}{\partial t}\,dV$. Equating these two quantities, we have

$$\iint_S \delta\mathbf{F}\cdot\mathbf{N}\,dS = -\iiint_D \frac{\partial\delta}{\partial t}\,dV$$

By the divergence theorem, it follows that

$$\iint_S \delta\mathbf{F}\cdot\mathbf{N}\,dS = \iiint_D \operatorname{div}\delta\mathbf{F}\,dV$$

so that

$$\iiint_D \operatorname{div}\delta\mathbf{F}\,dV + \iiint_D \frac{\partial\delta}{\partial t}\,dV = 0$$

$$\iiint_D \left[\operatorname{div}\delta\mathbf{F} + \frac{\partial\delta}{\partial t}\right]dV = 0$$

This equation must hold for any region D, no matter how small, which means that the integrand of the integral must be 0. That is,

$$\operatorname{div}\delta\mathbf{F} = -\frac{\partial\delta}{\partial t} \qquad \blacksquare$$

It is known that the total heat contained in a body with uniform density δ and specific heat σ is $\iiint_D \sigma\delta T\,dV$, where T is the temperature. Thus, the amount of heat leaving D per unit of time is given by the derivative

$$-\frac{\partial}{\partial t}\left[\iiint_D \sigma\delta T\,dV\right] = \iiint_D -\sigma\delta\frac{\partial T}{\partial t}\,dV$$

In the following example, we use this result to obtain an important formula from mathematical physics.

EXAMPLE 5 Derivation of the heat equation

Let $T(x, y, z, t)$ be the temperature at each point (x, y, z) in a solid body D at time t. Given that the velocity of heat flow in the body is $\mathbf{F} = -K\,\nabla T$ for a positive constant K (called the **thermal conductivity**), show that

$$\frac{\partial T}{\partial t} = \frac{K}{\sigma\delta}\nabla^2 T$$

where σ is the specific heat of the body and δ is its density.

Solution

Let S be the closed surface that bounds D. Because $\mathbf{F}$ is the velocity of heat flow, the amount of heat leaving D per unit time is $\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$, and the divergence theorem applies:

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_D \operatorname{div}(-K\nabla T)\,dV = \iiint_D (-K\nabla\cdot\nabla T)\,dV$$
$$= \iiint_D -K\nabla^2 T\,dV$$

Since this is the amount of heat leaving D per unit of time, it must equal the heat integral from physics derived just before this example. Thus,

$$\iiint_D -K\nabla^2 T\,dV = \iiint_D -\sigma\delta\frac{\partial T}{\partial t}\,dV$$

This equation holds not only for the body as a whole, but for every part of the body, no matter how small. Thus, we can shrink the body to a single point, and it then can be shown that when this occurs the integrands are equal; that is,

$$-K\nabla^2 T = -\sigma\delta\frac{\partial T}{\partial t}$$
$$\frac{\partial T}{\partial t} = \frac{K}{\sigma\delta}\nabla^2 T \qquad \blacksquare$$

Recall from Section 13.4 that the *normal derivative* $\partial g/\partial n$ of a scalar function g defined on the closed surface S is the directional derivative of g in the direction of the outward unit normal vector $\mathbf{N}$ to S; that is,

$$\frac{\partial g}{\partial n} = \nabla g\cdot\mathbf{N}$$

We will use this equation in the following example, which involves a generalization of a property we first obtained for $\mathbb{R}^2$ in Example 6 of Section 13.4.

EXAMPLE 6 Derivation of Green's first identity

Show that if f and g are scalar functions such that $\mathbf{F} = f\nabla g$ is continuously differentiable in the solid domain D bounded by the closed surface S, then

$$\iiint_D [f\nabla^2 g + \nabla f\cdot\nabla g]\,dV = \iint_S f\frac{\partial g}{\partial n}\,dS$$

This is called **Green's first identity**.

Solution

We will apply the divergence theorem to the vector field $\mathbf{F}$ (note that $\mathbf{F}$ is continuously differentiable), but first we need to express div $\mathbf{F}$ in a more useful form.

$$
\begin{aligned}
\operatorname{div}(f\nabla g) &= \nabla\cdot(f\nabla g)\\
&= \left[\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right]\cdot\left[f\frac{\partial g}{\partial x}\mathbf{i}+f\frac{\partial g}{\partial y}\mathbf{j}+f\frac{\partial g}{\partial z}\mathbf{k}\right]\\
&= \frac{\partial}{\partial x}\left[f\frac{\partial g}{\partial x}\right]+\frac{\partial}{\partial y}\left[f\frac{\partial g}{\partial y}\right]+\frac{\partial}{\partial z}\left[f\frac{\partial g}{\partial z}\right]\\
&= \left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial x^2}\right]+\left[\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial y^2}\right]+\left[\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}+f\frac{\partial^2 g}{\partial z^2}\right]\\
&= \left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right]+f\left[\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right]\\
&= (\nabla f)\cdot(\nabla g)+f\nabla^2 g
\end{aligned}
$$

This calculation gives us the first step in the following computation.

$$
\begin{aligned}
\iiint_D [f\nabla^2 g+\nabla f\cdot\nabla g]\,dV &= \iiint_D \operatorname{div}(f\nabla g)\,dV\\
&= \iint_S (f\nabla g)\cdot\mathbf{N}\,dS && \text{Divergence theorem}\\
&= \iint_S f(\nabla g\cdot\mathbf{N})\,dS\\
&= \iint_S f\frac{\partial g}{\partial n}\,dS && \text{Because } \nabla g\cdot\mathbf{N}=\frac{\partial g}{\partial n} \text{ by definition}
\end{aligned}
$$

■

PHYSICAL INTERPRETATION OF DIVERGENCE

In Section 13.6, we used Stokes' theorem to give an interpretation of the curl as a measure of the tendency of a fluid to swirl (the circulation). Our last example gives an analogous interpretation of divergence. In particular, we show that the net rate of fluid mass flowing away (that is, "diverging") from point P_0 is given by div $\mathbf{F}_0$. This is the reason P_0 is a source if div $\mathbf{F}_0 > 0$ (mass flowing out from P_0) and a sink if div $\mathbf{F}_0 < 0$ (mass flowing back into P_0).

EXAMPLE 7 Physical interpretation of divergence

Let $\mathbf{F} = \delta\,\mathbf{V}$ be the flux density associated with a fluid of density δ flowing with velocity $\mathbf{V}$ and let P_0 be a point inside a solid region where the conditions of the divergence theorem are satisfied. Prove that

$$\operatorname{div}\mathbf{F}_0 = \lim_{r\to 0}\frac{1}{V(r)}\iint_{S(r)}\mathbf{F}\cdot\mathbf{N}\,dS$$

where div $\mathbf{F}_0$ denotes the value of div $\mathbf{F}$ at P_0, and $S(r)$ is a sphere centered at P_0 with volume $V(r) = \frac{4}{3}\pi r^3$.

Solution

Applying the divergence theorem to the solid sphere (ball) $B(r)$ with surface $S(r)$, we obtain

$$\iint_{S(r)}\mathbf{F}\cdot\mathbf{N}\,dS = \iiint_{B(r)}\operatorname{div}\mathbf{F}\,dV$$

The mean value theorem (for triple integrals) tells us that

$$\frac{1}{V(r)}\iiint\limits_{B(r)} \operatorname{div}\mathbf{F}\,dV = \operatorname{div}\mathbf{F}^*$$

where div $\mathbf{F}^*$ denotes the value of div $\mathbf{F}$ at some point P^* in the ball $B(r)$. Combining these results, we find that

$$\iint\limits_{S(r)} \mathbf{F}\cdot\mathbf{N}\,dS = \iiint\limits_{B(r)} \operatorname{div}\mathbf{F}\,dV = V(r)\operatorname{div}\mathbf{F}^*$$

or

$$\frac{1}{V(r)}\iint\limits_{S(r)} \mathbf{F}\cdot\mathbf{N}\,dS = \operatorname{div}\mathbf{F}^*$$

Since the point P^* is inside the ball $B(r)$ centered at P_0, it follows that $P^* \to P_0$ as $r \to 0$, so div $\mathbf{F}^* \to$ div $\mathbf{F}_0$ and we have

$$\lim_{r\to 0}\frac{1}{V(r)}\iint\limits_{S(r)} \mathbf{F}\cdot\mathbf{N}\,dS = \lim_{r\to 0}\operatorname{div}\mathbf{F}^* = \operatorname{div}\mathbf{F}_0$$

as claimed. ■

HW③ pg146 - pg154)

Problems

13.7 PROBLEM SET

Q 4, 6, 7, 9, 12, 14, 16

A *Verify the divergence theorem for the vector function* $\mathbf{F}$ *and solid* D *given in Problems 1–4. Assume* $\mathbf{N}$ *is the unit normal vector pointing away from the origin.*

1. $\mathbf{F} = xz\mathbf{i} + y^2\mathbf{j} + 2z\mathbf{k}$; D is the ball $x^2 + y^2 + z^2 \le 4$.
2. $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j}$; D is the interior of the paraboloid $z = x^2 + y^2$, $0 \le z \le 9$.
3. $\mathbf{F} = 2y^2\mathbf{j}$; D is the tetrahedron bounded by the coordinate planes and the $x + 4y + z = 8$.
4. $\mathbf{F} = 3x\mathbf{i} + 5y\mathbf{j} + 6z\mathbf{k}$; D is the tetrahedron bounded by the coordinate planes and the plane $2x + y + z = 4$. 224/3

Use the divergence theorem in Problems 5–19 to evaluate the surface integral $\iint\limits_S \mathbf{F}\cdot\mathbf{N}\,dS$ *for the given choice of* $\mathbf{F}$ *and closed boundary surface* S. *Assume* $\mathbf{N}$ *is the outward unit normal vector field.*

5. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.
6. $\mathbf{F} = xyz\mathbf{j}$; S is the cylinder $x^2 + y^2 = 9$, for $0 \le z \le 5$. 0
7. $\mathbf{F} = (\cos yz)\mathbf{i} + e^{xz}\mathbf{j} + 3z^2\mathbf{k}$; S is the hemisphere $z = \sqrt{4 - x^2 - y^2}$ together with the disk $x^2 + y^2 \le 4$ in the xy-plane. 24π
8. $\mathbf{F} = \text{curl}[e^{xz}\mathbf{i} - 4\mathbf{j} + (\sin xyz)\mathbf{k}]$; S is the ellipsoid $2x^2 + 3y^2 + 7z^2 = 1$.
9. $\mathbf{F} = (x^2 + y^2 - z^2)\mathbf{i} + x^2y\mathbf{j} + 3z\mathbf{k}$; S is the surface comprised of the five faces of the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$, missing $z = 0$. 13/3
10. $\mathbf{F} = 2y\mathbf{i} - z\mathbf{j} + 3x\mathbf{k}$; S is the surface comprised of the five faces of the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$, missing $z = 0$.
11. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the paraboloid $z = x^2 + y^2$ for $0 \le z \le 9$.
12. $\mathbf{F} = \text{curl}(y\mathbf{i} + x\mathbf{j} - z\mathbf{k})$; S is the hemisphere $z = \sqrt{4 - x^2 - y^2}$ together with the disk $x^2 + y^2 \le 4$ in the xy-plane. 0 div(curl F)=0
13. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$; S is the sphere $x^2 + y^2 + z^2 = 4$.
14. $\mathbf{F} = xyz\mathbf{i} + xyz\mathbf{j} + xyz\mathbf{k}$; S is the surface of the box $0 \le x \le 1$, $0 \le y \le 2, 0 \le z \le 3$. 33/2
15. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z^2 - 1)\mathbf{k}$; S is the surface of a solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 1$.
16. $\mathbf{F} = (x^5 + 10xy^2z^2)\mathbf{i} + (y^5 + 10yx^2z^2)\mathbf{j} + (z^5 + 10zx^2y^2)\mathbf{k}$; S is the closed hemispherical surface $z = \sqrt{1 - x^2 - y^2}$ together with the disk $x^2 + y^2 \le 1$ in the xy-plane. 10π/7
17. $\mathbf{F} = xy^2\mathbf{i} + yz^2\mathbf{j} + x^2y\mathbf{k}$; S is the surface bounded above by the sphere $\rho = 2$ and below by the cone $\phi = \frac{\pi}{4}$ (in spherical coordinates). *Note*: S is the surface of an "ice cream cone."
18. $\mathbf{F} = xy^2\mathbf{i} + yz^2\mathbf{j} + x^2z\mathbf{k}$; S is the surface bounded above by the sphere $\rho = 2$ and below by the cone $\phi = \frac{\pi}{4}$ (in spherical coordinates). *Note*: S is the surface of an "ice cream cone."

19. $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + 3a^2z\mathbf{k}$ (constant $a > 0$); S is the surface bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = 0$ and $z = 1$.

B 20. Suppose that S is a closed surface that encloses a solid region D.

a. Show that the volume of D is given by

$$V(D) = \frac{1}{3}\iint_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{N}\,dS$$

where $\mathbf{N}$ is an outward unit normal vector to S.

b. Use the formula in part a to find the volume of the hemisphere

$$z = \sqrt{R^2 - x^2 - y^2}$$

21. Use the divergence theorem to evaluate

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$$\iint_S \|\mathbf{R}\|\mathbf{R} \cdot \mathbf{N}\,dS$$

where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = a^2$, with constant $a > 0$. $4\pi a^4$

22. **Counterexample Problem** Let $\mathbf{F} = \langle f(y, z), g(x, z), h(x, y)\rangle$ and let S be the surface of a solid G. Either prove that the flux of $\mathbf{F}$ across S is zero or find a counterexample.

23. **Exploration Problem** Let $f(x, y, z)$ be a differentiable nonzero scalar function. State an additional property of f that will guarantee

$$\iint_S f\nabla f \cdot \mathbf{N}\,dS = \iiint_G \|\nabla f\|^2\,dV$$

for any solid region G bounded by the closed oriented surface S.

24. The moment of inertia about the z-axis of a solid D of constant density $\delta = a$ is given by

$$I_z = \iiint_T a(x^2 + y^2)\,dV$$

Express this integral as a surface integral over the surface S that bounds D.

C 25. Let u be a scalar function with continuous second partial derivatives in a region containing the solid region D, with closed boundary surface S.

a. Show that $\displaystyle\iint_S \frac{\partial u}{\partial n}\,dS = \iiint_D \nabla^2 u\,dV$.

b. Let $u = x + y + z$ and $v = \frac{1}{2}(x^2 + y^2 + z^2)$. Evaluate $\displaystyle\iint_S (u\nabla v) \cdot \mathbf{N}\,dS$, where S is the boundary of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

26. Let f and g be scalar functions such that $\mathbf{F} = f\nabla g$ is continuously differentiable in the region D, which is bounded by the closed surface S. Prove *Green's second identity* using the divergence theorem:

$$\iiint_D (f\nabla^2 g - g\nabla^2 f)\,dV = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS$$

27. Show that if g is harmonic in the region D, then

$$\iint_S \frac{\partial g}{\partial n}\,dS = 0$$

where the closed surface S is the boundary of D. (Recall that g harmonic means $\nabla^2 g = 0$.)

28. Show that $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{N}\,dS = 0$ if S is a closed surface and $\mathbf{F} = \operatorname{curl} \mathbf{U}$ throughout the interior of S for some vector field $\mathbf{U}$ with continuous second partial derivatives. A vector field $\mathbf{U}$ with this property is said to be a **vector potential** for $\mathbf{F}$.

29. In our derivation of the heat equation in this section, we assumed that the coefficient of thermal conductivity K is constant (no sinks or sources). If $K = K(x, y, z)$ is a variable, show that the heat equation becomes

$$K\nabla^2 T + \nabla K \cdot \nabla T = \sigma\delta\frac{\partial T}{\partial t}$$

30. An electric charge q located at the origin produces the electric field

$$\mathbf{E} = \frac{q\mathbf{R}}{4\pi\epsilon\|\mathbf{R}\|^3}$$

where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and ϵ is a physical constant, called the **electric permittivity**.

a. Show that

$$\iint_S \mathbf{E} \cdot \mathbf{N}\,dS = 0$$

if the closed surface S does not enclose the origin. This is **Gauss' law**.

b. Show that

$$\iint_S \mathbf{E} \cdot \mathbf{N}\,dS = \frac{q}{\epsilon}$$

in the case where the closed surface S encloses the origin. Note that the divergence theorem does not apply directly to this case.

31. Gauss' law can be expressed as

$$\iint_S \mathbf{D} \cdot \mathbf{N}\,dS = q$$

where $\mathbf{D} = \epsilon\mathbf{E}$ is the electric flux density, with electric intensity $\mathbf{E}$, permittivity ϵ, and q a constant. Show that div $\mathbf{D} = Q$, where Q is the charge density; that is,

$$\iiint_V Q\,dV = q$$

32. **HISTORICAL QUEST** James Clerk Maxwell was one of the greatest physicists of all time. Using the experimental discoveries of Michael Faraday as a basis, he was able to express the governing rules for electrical and magnetic fields in precise mathematical form. In 1871, he published his *Theory of Heat and Magnetism*, which formed the basis for modern electromagnetic theory and contributed to quantum theory and special relativity. Maxwell was influential in convincing other mathematicians and scientists to use vectors and was interested in areas as diverse as the behavior of light and the statistical behavior of molecular motion. He was sometimes referred to as dp/dt

JAMES CLERK MAXWELL 1831–1879

because in thermodynamics, $dp/dt = JCM$, a unit of measurement named for him. ■

For this Quest you are to derive **Maxwell's equation for the electric intensity E:**

$$(\nabla \cdot \nabla)\mathbf{E} = \mu\sigma\frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

To derive this equation, you need to know that $\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\operatorname{curl} \mathbf{H} = \sigma\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t}$ where **E** is electric intensity, **B** is magnetic flux density, **H** is magnetic intensity, and σ, ϵ, and μ are positive constants.

a. Use the fact that $\mathbf{B} = \mu\mathbf{H}$ to show that

$$\operatorname{curl}(\operatorname{curl} \mathbf{E}) = -\mu\frac{\partial}{\partial t}(\operatorname{curl} \mathbf{H})$$

b. Next, show that for any vector field $\mathbf{F} = \langle f, g, h\rangle$,

$$\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \nabla(\operatorname{div} \mathbf{F}) - \nabla \cdot \nabla\mathbf{F}$$

c. Use the formula in **b** to show that

$$\nabla(\operatorname{div} \mathbf{E}) - (\nabla \cdot \nabla)\mathbf{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t}\right)$$

d. Complete the derivation of *Maxwell's electric intensity equation*, assuming that the charge density Q is 0 so that $\operatorname{div} \mathbf{E} = 0$. (See Problem 31.)

CHAPTER 13 REVIEW

Chapter Checklist

In the following statements, assume that all required conditions are satisfied. Main results are collected here (without hypotheses) so that you can compare and contrast various results and conclusions.

Scalar function: $f(x, y, z)$

Vector field: $\mathbf{F}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$

Notation:

Del operator: $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$

Gradient: $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$

Laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = f_{xx} + f_{yy} + f_{zz}$

Normal derivative: $\frac{\partial f}{\partial n} = \nabla f \cdot \mathbf{N}$

Derivatives of a vector field:

$\operatorname{div} \mathbf{F} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ This is a scalar derivative.

$= \nabla \cdot \mathbf{F}$

If P is a point (x_0, y_0, z_0), then P is a

source if $\operatorname{div} \mathbf{F} > 0$

sink if $\operatorname{div} \mathbf{F} < 0$

F is incompressible if $\operatorname{div} \mathbf{F} = 0$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$ This is a vector derivative.

$= \nabla \times \mathbf{F}$

F is irrotational if $\operatorname{curl} \mathbf{F} = \mathbf{0}$

Line Integral: If $f(x, y, z)$ is defined on the smooth curve C with parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, then the line integral of f over C is given by

$$\int_C f(x, y, z)\,ds = \int_a^b f[x(t), y(t), z(t)]\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

Green's Theorem expresses an important relationship between a line integral over a Jordan (simple closed) curve in the plane and a double integral over the region bounded by the curve. Let D be a simply connected region with a positively oriented piecewise smooth boundary C. Then if the vector field $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is continuously differentiable on D, we have

$$\oint_C (M\,dx + N\,dy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Conservative Vector Fields and Path Independence: Let **F** be a continuous vector field on the open connected set D. Then the following three conditions are either all true or all false:

a. **F** is conservative on D; that is, $\mathbf{F} = \nabla f$ for some function f defined on D.

b. $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$ for every piecewise smooth closed curve in D.

c. $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path within D.

Also, if D is simply connected (no holes), then **F** is conservative if and only if $\operatorname{curl} \mathbf{F} = \mathbf{0}$ in D.

Fundamental Theorem for Line Integrals: Let C be a piecewise smooth curve that is parameterized by the vector function $\mathbf{R}(t)$ for $a \le t \le b$. If f is a differentiable function of two or three variables whose gradient $\mathbf{F} = \nabla f$ is continuous on C, then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = f(Q) - f(P)$$

where $Q = \mathbf{R}(b)$ and $P = \mathbf{R}(a)$ are the endpoints of C.

EVALUATION OF LINE INTEGRALS: $\int_C \mathbf{F} \cdot d\mathbf{R}$

Step 1. Check to see whether **F** is conservative; if it is, then

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = 0 \quad \text{if } C \text{ is closed}$$

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = f(Q) - f(P) \quad \text{Initial point } P, \text{ terminal point } Q$$

Step 2. If **F** is not conservative, and C is a closed curve bounding a surface S, use Stokes' theorem (or Green's theorem in $\mathbb{R}^2$) to equate the given integral to a surface integral, namely,

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{N})\, dS$$

Step 3. As a last resort, parameterize **F** and **R**. Let $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ for $a \le t \le b$.

$$\int_C f(x, y, z)\, dx = \int_a^b f[x(t), y(t), z(t)] \frac{dx}{dt}\, dt$$

$$\int_C f(x, y, z)\, ds$$

$$= \int_a^b f[x(t), y(t), z(t)] \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

Surface Integral: Let S be a surface defined by $z = f(x, y)$ and D its projection on the xy-plane. If f, f_x, and f_y are continuous in D and g is continuous on S, then the surface integral of g over S is

$$\iint_S g(x, y, z)\, dS$$

$$= \iint_D g(x, y, f(x, y)) \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

Stokes' theorem: Let S be an oriented surface with unit normal vector field **N**, and assume that S is bounded by a piecewise smooth Jordan curve C whose orientation is compatible with that of S. If **F** is a vector field that is continuously differentiable on S, then

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{N})\, dS$$

The divergence theorem: Let S be a smooth, orientable surface that encloses a solid region D in $\mathbb{R}^3$. If **F** is a continuous vector field whose components have continuous partial derivatives in an open set containing D, then

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV$$

where **N** is the outward unit normal field for the surface S.

EVALUATION OF FLUX INTEGRALS: $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$

Step 1. If the surface S is a closed surface bounding the solid region D, use the divergence theorem to write the flux integral as a triple integral.

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV$$

Step 2. If step 1 does not apply, parameterize **F**, **N**, and dS. In the special case where S has the form $z = f(x, y)$, we have

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_D \mathbf{F}(x, y, f(x, y)) \cdot (-f_x, -f_y, 1)\, dA$$

for an upward normal and

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_D \mathbf{F}(x, y, f(x, y)) \cdot (f_x, f_y, -1)\, dA$$

for a downward normal.

Proficiency Examination

CONCEPT PROBLEMS

1. What is a vector field?
2. What is the divergence of a vector field?
3. What is the curl of a vector field?
4. What is the del operator?
5. What is Laplace's equation?
6. What is the difference between the Riemann integral and a line integral? Discuss.
7. What is the formula for a line integral of a vector field?
8. How do we find work as a line integral?
9. What is the formula for a line integral in terms of arc length parameter?
10. State the fundamental theorem for line integrals.
11. Define a conservative vector field.
12. What is the scalar potential of a conservative vector field?
13. What is a Jordan curve?
14. State Green's theorem.
15. How can you use Green's theorem to find area as a line integral?
16. What is a normal derivative?
17. Define a surface integral.
18. What is the formula for a surface integral of a surface defined parametrically?
19. What is a flux integral?
20. State Stokes' theorem.
21. State the conservative vector field theorem.
22. State the divergence theorem.

PRACTICE PROBLEMS

23. Show that $yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ is conservative and find a scalar potential function.
24. Compute $\operatorname{div} \mathbf{F}$ and $\operatorname{curl} \mathbf{F}$ for $\mathbf{F} = x^2y\mathbf{i} - e^{yz}\mathbf{j} + \frac{1}{2}x\mathbf{k}$.

25. Use Green's theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = (2x + y)\mathbf{i} + 3y^2\mathbf{j}$ and C is the boundary of the triangle T with vertices $(-1, 2)$, $(0, 0)$, $(1, 2)$, traversed in the given order.

26. Use Stokes' theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = 2y\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ and C is the intersection of the plane $z = x + 2$ and the sphere $x^2 + y^2 + z^2 = 4z$, traversed counterclockwise as viewed from above.

27. Use the divergence theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$, where $\mathbf{F} = x^2\mathbf{i} + (y + z)\mathbf{j} - 2z\mathbf{k}$, and S is the surface of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

28. Evaluate $\oint_C \dfrac{x\,dx + y\,dy}{(x^2 + y^2)^2}$, where C is the path shown in Figure 13.51, traversed counterclockwise.

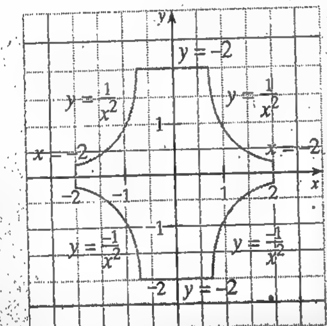


Figure 13.51 Curve C

29. An object with mass m travels counterclockwise (as viewed from above) in the circular orbit $x^2 + y^2 = 9$, $z = 2$, with angular speed ω. The mass is subject to a centrifugal force $\mathbf{F} = m\omega^2\mathbf{R}$, where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\mathbf{F}$ is conservative, and find a scalar potential function for $\mathbf{F}$.

30. Find the work done as an object moves in the force field given in Problem 29 from point $(3, 0, 2)$ to $(-3, 0, 2)$ along the circle $x^2 + y^2 = 9$ in the plane $z = 2$.

Supplementary Problems

In Problems 1–6, determine whether the given vector field is conservative, and if it is, find a scalar potential function.

1. $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j}$
2. $\mathbf{F} = xy^{-2}\mathbf{i} + x^{-2}y\mathbf{j}$
3. $\mathbf{F} = y^{-3}\mathbf{i} + (-3xy^{-4} + \cos y)\mathbf{j}$
4. $\mathbf{F} = y^2\mathbf{i} + (2xy)\mathbf{j}$
5. $\mathbf{F} = \left(\dfrac{1}{y} + \dfrac{y}{x^2}\right)\mathbf{i} - \left(\dfrac{x}{y^2} - \dfrac{1}{x}\right)\mathbf{j}$
6. $\mathbf{F} = \left[2x \tan^{-1}\left(\dfrac{y}{x}\right) - y\right]\mathbf{i} + \left[2y \tan^{-1}\left(\dfrac{y}{x}\right) + x\right]\mathbf{j}$

In Problems 7–12, find $\int_C \mathbf{F} \cdot d\mathbf{R}$, where C is the curve $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j}$, $1 \le t \le 2$. Note that these are the same as the vector fields in Problems 1–6.

7. $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j}$
8. $\mathbf{F} = xy^{-2}\mathbf{i} + x^{-2}y\mathbf{j}$
9. $\mathbf{F} = y^{-3}\mathbf{i} + (-3xy^{-4} + \cos y)\mathbf{j}$
10. $\mathbf{F} = y^2\mathbf{i} + (2xy)\mathbf{j}$
11. $\mathbf{F} = \left(\dfrac{1}{y} + \dfrac{y}{x^2}\right)\mathbf{i} - \left(\dfrac{x}{y^2} - \dfrac{1}{x}\right)\mathbf{j}$
12. $\mathbf{F} = \left[2x \tan^{-1}\left(\dfrac{y}{x}\right) - y\right]\mathbf{i} + \left[2y \tan^{-1}\left(\dfrac{y}{x}\right) + x\right]\mathbf{j}$

In Problems 13–16, find div $\mathbf{F}$ and curl $\mathbf{F}$.

13. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
14. $\mathbf{F} = \left(\tan^{-1}\dfrac{y}{x}\right)\mathbf{i} - 3\mathbf{j} + z^2\mathbf{k}$
15. $\mathbf{F} = \dfrac{1}{r}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, where $r = \sqrt{x^2 + y^2 + z^2}$, $r \ne 0$
16. $\mathbf{F} = (xy \sin z)\mathbf{i} + (x^2 \cos yz)\mathbf{j} + (z \sin xy)\mathbf{k}$

Evaluate the line integrals in Problems 17–18 by parameterization.

17. $\int_C [(\sin \pi y)\, dx + (\cos \pi x)\, dy]$, where C is the line segment from $(1, 0)$ to $(\pi, 0)$, followed by the line segment from $(\pi, 0)$ to (π, π)
18. $\int_C (z\,dx - x\,dy + dz)$, where C is the arc of the helix $x = 3\sin t$, $y = 3\cos t$, $z = t$ for $0 \le t \le \frac{\pi}{4}$

In Problems 19–33, evaluate the line integral or the surface integral. In each surface integral, assume that $\mathbf{N}$ is the outward unit normal vector field.

19. $\int_C [yz\,dx + xz\,dy + (xy + 2)\,dz]$, where C is the curve $\mathbf{R}(t) = (\tan^{-1} t)\mathbf{i} + t^2\mathbf{j} - 3t\mathbf{k}$, $0 \le t \le 1$
20. $\int_C [(x^2 + y)\,dx + xz\,dy - (y + z)\,dz]$, where C is the curve $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}$, $0 \le t \le 1$
21. $\iint_S (3x^2 + y - 2z)\,dS$, where S is the surface $\mathbf{R}(u, v) = u\mathbf{i} + (u + v)\mathbf{j} + u\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le 1$
22. $\iint_S \mathbf{F} \cdot \mathbf{N}\,dS$, where $\mathbf{F} = 3x\mathbf{i} + z^2\mathbf{j} - 2y\mathbf{k}$ and S is the surface of the hemisphere $z = \sqrt{4 - x^2 - y^2}$
23. $\int_C (x\,dx + x\,dy - y\,dz)$, where C is the curve $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$
24. $\int_C (x^2\,dx - 3y^2\,dz)$, where C is the line segment from $(0, 1, 1)$ to $(1, 1, 2)$
25. $\int_C (x^2\,dx + y\,dy)$, where C is the curve $\mathbf{R}(t) = (t \sin t)\mathbf{i} + (1 - t\cos t)\mathbf{j}$, $0 \le t \le 2\pi$

26. $\oint_C (xy\,dx - x^2\,dy)$, where C is the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$ traversed counterclockwise

27. $\oint_C (y\,dx + x\,dy - 2\,dz)$, where C is the curve of intersection of the cylinder $x^2 + y^2 = 2x$ and the plane $x = z$, traversed counterclockwise as viewed from above

28. $\oint_C [(y + z)\,dx + (x + z)\,dy + (x + y)\,dz]$, where C is the curve of intersection of the sphere $x^2 + (y-3)^2 + z^2 = 9$ and the plane $x + 2y + z = 3$, traversed counterclockwise as viewed from above

29. $\oint_C (-2y\,dx + 2x\,dy + dz)$, where C is the circle $x^2 + y^2 = 1$ in the plane $z = 3$, traversed counterclockwise as viewed from above

30. $\iint_S (\text{curl } y\mathbf{i}) \cdot \mathbf{N}\,dS$, where S is the hemisphere $z = \sqrt{1 - x^2 - y^2}$

31. $\iint_S (2x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}) \cdot \mathbf{N}\,dS$, where S is the surface of the ellipsoid $2x^2 + y^2 + z^2 = 1$

32. $\iint_S (y^2\mathbf{i} + y^2\mathbf{j} + yz\mathbf{k}) \cdot \mathbf{N}\,dS$ where S is the surface of the tetrahedron bounded by the plane $2x + 3y + z = 1$ and the coordinate planes, in the first octant

33. $\iint_S \nabla\phi \cdot \mathbf{N}\,dS$, where $\phi(x, y, z) = 2x + 3y$ and S is the portion of the plane $ax + by + cz = 1$ $(a > 0, b > 0, c > 0)$ that lies in the first octant

In Problems 34–38 find $\iint_S \mathbf{F} \cdot \mathbf{N}\,dS$. *Assume* $\mathbf{N}$ *is the outward unit normal vector field for* S.

34. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$, and S is the surface of the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

35. $\mathbf{F} = 2yz\mathbf{i} + (\tan^{-1} xz)\mathbf{j} + e^{xy}\mathbf{k}$, and S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

36. $\mathbf{F} = x\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$, and S is the paraboloid $y = x^2 + z^2$ with $x^2 + z^2 < 9$. The disk $x^2 + z^2 = 9$ is omitted; that is, the paraboloid is open on the right.

37. $\mathbf{F} = xyz\mathbf{i} + xyz\mathbf{j} + xyz\mathbf{k}$, and S is the surface of the five faces of the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 < z \le 1$, missing $z = 0$.

38. $\mathbf{F} = xy\mathbf{i} - 2z\mathbf{j}$, and S is the surface defined parametrically by $\mathbf{R}(u, v) = u\mathbf{i} + v\mathbf{j} + u\mathbf{k}$ for $0 \le u \le 1, 0 \le v \le 1$.

Find all real numbers c for which each vector field in Problems 39–41 is conservative.

39. $\mathbf{F}(x, y) = (\sqrt{x} + 3xy)\mathbf{i} + (cx^2 + 4y)\mathbf{j}$

40. $\mathbf{F}(x, y) = \left(\frac{cy}{x^3} + \frac{y}{x^2}\right)\mathbf{i} + \left(\frac{1}{x^2} - \frac{1}{x}\right)\mathbf{j}$

41. $\mathbf{F}(x, y, z) = e^{yz/x}\left[\left(\frac{cyz}{x^2}\right)\mathbf{i} + \left(\frac{z}{x}\right)\mathbf{j} + \left(\frac{y}{x}\right)\mathbf{k}\right]$

42. Let $\mathbf{F} = (y^2 + x^{-2}ye^{x/y})\mathbf{i} + (2xy + z - x^{-1}e^{x/y})\mathbf{j} + y\mathbf{k}$. Is $\mathbf{F}$ conservative?

43. Find the work done when an object moves in the force field $\mathbf{F} = 2x\mathbf{i} - (x + z)\mathbf{j} + (y - x)\mathbf{k}$ along the path given by $\mathbf{R}(t) = t^2\mathbf{i} + (t^2 - t)\mathbf{j} + 3\mathbf{k}$, $0 \le t \le 1$.

44. Show that the force field $\mathbf{F} = yz^2\,\mathbf{i} + (xz^2 - 1)\mathbf{j} + (2xyz - 1)\mathbf{k}$ is conservative, and determine the work done when an object moves in the force field from the origin to the point $(1, 0, 1)$.

45. Find a region R in the plane where the vector field $\mathbf{F} = \frac{1}{x + y}(\mathbf{i} + \mathbf{j})$ is conservative. Then evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$, where C is any path in R from the point $P_0(a, b)$ to $P_1(c, d)$.

46. If u is a scalar function and $\mathbf{F}$ is a continuously differentiable vector field, show that $\text{curl}(u\mathbf{F}) = u\,\text{curl}\,\mathbf{F} + (\nabla u \times \mathbf{F})$.

47. Show that $\text{div}\,(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}\,\mathbf{F} - \mathbf{F} \cdot \text{curl}\,\mathbf{G}$, for any continuously differentiable vector fields $\mathbf{F}$ and $\mathbf{G}$.

48. A vector field $\mathbf{F}$ is *incompressible* in a region D if $\text{div}\,\mathbf{F} = 0$ throughout D. If $\mathbf{F}$ and $\mathbf{G}$ are both conservative vector fields in D, show that $\mathbf{F} \times \mathbf{G}$ is incompressible.

49. If $\mathbf{F} = \text{curl}\,\mathbf{G}$, show that $\mathbf{F}$ is incompressible. (See Problem 48.)

50. Suppose $\mathbf{F} = f(x, y, z)\,\mathbf{A}$, where $\mathbf{A}$ is a constant vector and f is a scalar function. Show that curl $\mathbf{F}$ is orthogonal to $\mathbf{A}$ and to ∇f.

51. If $\mathbf{A}$ is a constant vector and $\mathbf{F}$ is a continuously differentiable vector field, show that $\text{div}\,(\mathbf{A} \times \mathbf{F}) = -\mathbf{A} \cdot \text{curl}\,\mathbf{F}$.

52. Evaluate the line integral $\int_C \frac{x\,dx - y\,dy}{x^2 - y^2}$, where C is any path in the xy-plane that is interior to the region $x > 0$, $y < x$, $y > -x$ and connects the point $(5, 4)$ to $(2, 0)$.

53. Evaluate $\oint_C \left(\frac{-y}{x^2}\,dx + \frac{1}{x}\,dy\right)$, where C is the closed path $(x - 2)^2 + y^2 = 1$, traversed once counterclockwise.

54. **Counterexample Problem** Let $u(x, y)$ and $v(x, y)$ be functions of two variables with continuous partial derivatives everywhere in the plane, and suppose that u and v satisfy the equation

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

for all (x, y). Are u and v necessarily harmonic? Either show that they are or find a counterexample.

55. **a.** Find a region in the plane where the vector field

$$\mathbf{F} = \left(\frac{1 + y^2}{x^3}\right)\mathbf{i} - \left(\frac{y + x^2y}{x^2}\right)\mathbf{j}$$

is conservative, and find a scalar potential for $\mathbf{F}$.

b. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$, where C is a path from $(1, 1)$ to $(3, 4)$.

Are there any limitations on the path C? Explain.

56. Determine the most general function $u(x, y)$ for which the vector field $\mathbf{F} = u(x, y)\mathbf{i} + (2ye^x + y^2e^{3x})\mathbf{j}$ will be conservative.

57. Consider the line integral $\int_C \left(\frac{dx}{y} + \frac{dy}{x}\right)$, where C is the closed triangular path formed by the lines $y = 2x$, $x + 2y = 5$, and $x = 2$, traversed counterclockwise. First evaluate the line

integral directly (by parameterizing C) and then by using Green's theorem.

58. If S is a closed surface in a region R and $\mathbf{F}$ is a twice continuously differentiable vector field on R, show that

$$\iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{N})\,dS = 0$$

where $\mathbf{N}$ is the outward unit normal vector to S.

59. A certain closed path C in the plane $2x+2y+z=1$ is known to project onto the unit circle $x^2+y^2=1$ in the xy-plane. Let c be a constant, and let $\mathbf{R}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Use Stokes' theorem to evaluate

$$\oint_C (c\mathbf{k}\times\mathbf{R})\cdot d\mathbf{R}$$

60. a. Show that if the scalar function w is harmonic, then

$$\nabla\cdot(w\nabla w)=\|\nabla w\|^2$$

b. Let $w=x-y+2z$, and let S be the surface of the sphere $x^2+y^2+z^2=9$. Evaluate

$$\iint_S w\frac{\partial w}{\partial n}\,dS$$

61. A particle moves along a curve C in space that is given parametrically by $\mathbf{R}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$ for $a\le t\le b$. A force field $\mathbf{F}$ is applied to the particle in such a way that $\mathbf{F}$ is always perpendicular to the path C. How much work is performed by the force field $\mathbf{F}$ when the particle moves from the point where $t=a$ to the point where $t=b$?

62. If $\mathbf{D}$ is the electric displacement field, then $\operatorname{div}\mathbf{D}=\phi$, where ϕ is the *charge density*. A region of space is said to be *charge-free* if $\phi=0$ there. Describe the charge-free regions of the electric displacement field $\mathbf{D}=2x^2\mathbf{i}+3y^2\mathbf{j}-2z^2\mathbf{k}$.

63. A satellite weighing 10,000 kg travels in a circular orbit 7,000 km from the center of the earth. How much work is done by gravity on the satellite during half a revolution?

64. If $\mathbf{F}$ is a conservative force field, the scalar function f such that $\mathbf{F}=-\nabla f$ is called the *potential energy* (see Problem 53, Section 13.3). Suppose an object with mass 10 g moves in the force field in such a way that its speed decreases from 3 cm/s to 2.5 cm/s. What is the corresponding change in the potential energy of the object?

65. Find the work done by the force field

$$\mathbf{F}=2xyz\mathbf{i}+\left(x^2z-\frac{1}{z}\tan^{-1}\frac{y}{z}\right)\mathbf{j}+\left(x^2y+\frac{y}{z^2}\tan^{-1}\frac{y}{z}\right)\mathbf{k}$$

in moving an object along the circular helix

$$d\mathbf{R}(t)=(\sin\pi t)\mathbf{i}+(\cos\pi t)\mathbf{j}+(2t+1)\mathbf{k}$$

for $0\le t\le\frac{1}{2}$.

66. Find the work done when an object moves against the force field $\mathbf{F}=4y^2\mathbf{i}+(3x+y)\mathbf{j}$ from $(1,0)$ to $(-1,0)$ along the top half of the ellipse $x^2+\dfrac{y^2}{k^2}=1$. Which value of k minimizes the work?

67. Evaluate the line integral

$$\oint_C \frac{-y\,dx+x\,dy}{x^2+y^2}$$

where C is the limaçon given in polar coordinates by $r=3+2\cos\theta$, $0\le\theta\le 2\pi$, traversed counterclockwise.

68. Suppose f and g are both harmonic in the region R with boundary surface S. Show that

a. $\displaystyle\iint_S f\frac{\partial g}{\partial n}\,dS=\iint_S g\frac{\partial f}{\partial n}\,dS$

b. $\displaystyle\iint_S f\frac{\partial f}{\partial n}\,dS=\iiint_D \|\nabla f\|^2\,dV$

69. Show that $\operatorname{curl}(\operatorname{curl}\mathbf{F})=\nabla(\operatorname{div}\mathbf{F})-\nabla^2\mathbf{F}$ if the components of $\mathbf{F}$ have continuous second-order partial derivatives.

70. Evaluate the surface integral $\displaystyle\iint_S \frac{\partial f}{\partial n}\,dS$, where S is the surface of the unit sphere $x^2+y^2+z^2=1$ and f is a scalar field such that $\|\nabla f\|^2=3f$ and $\operatorname{div}(f\nabla f)=7f$.

71. Evaluate the surface integral $\displaystyle\iint_S dS$, where S is the torus

$$\mathbf{R}(u,v)=[(a+b\cos v)\cos u]\mathbf{i}+[(a+b\cos v)\sin u]\mathbf{j}+(b\sin v)\mathbf{k}$$

for $0<b<a$ and $0\le u\le 2\pi$, $0\le v\le 2\pi$.

72. Evaluate $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$, where $\mathbf{F}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and S is the closed cubic surface with a corner block removed, as shown in Figure 13.52.

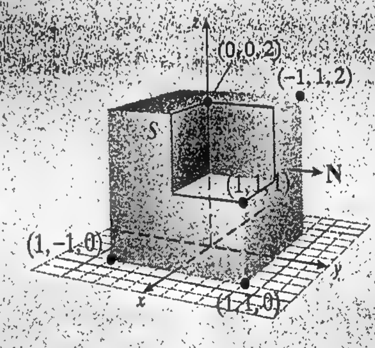


Figure 13.52 Cube with a corner removed

73. Counterexample Problem Prove or disprove that it does not matter which corner is removed in Problem 72.

74. Show that a lamina that covers a standard region D in the plane with density $\delta=1$ has moment of inertia

$$I=\frac{1}{3}\oint_C(-y^3\,dx+x^3\,dy)$$

with respect to the z-axis, where C is the boundary curve of D.

75. Use vector analysis to find the centroid of the conical surface $z=\sqrt{x^2+y^2}$ between $z=0$ and $z=3$.

76. Putnam Examination Problem A force acts on the element ds of a closed plane curve. The magnitude of this force is $r^{-1}\,ds$, where r is the radius of curvature at the point considered, and the direction of the force is perpendicular to the curve; it points to the convex side. Show that the system of such forces acting on all elements of the curve keeps it in equilibrium.

Guest Essay

Continuous versus Discrete Mathematics

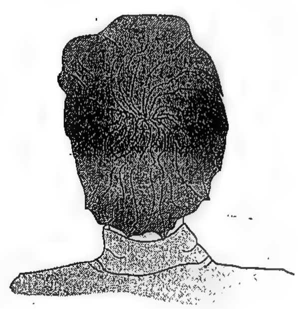
William F. Lucas is professor emeritus of mathematics and former department chairman at The Claremont Graduate University. He is known for his research in game theory, which provides a mathematical approach to the study of conflict, cooperation, and fairness. Dr. Lucas has also been active in educational reform efforts with a goal toward introducing more recently discovered topics into the mathematics curriculum.

Mathematicians often distinguish between *discrete* and *continuous* mathematics. The latter is illustrated by the calculus and its many descendants, including the subject of differential equations. Continuous mathematics deals with "solid" infinite sets such as the real number line $\mathbb{R}$ and functions defined on $\mathbb{R}$. One of the primary concerns is with the solutions of differential equations, which are typical families of such solid curves or surfaces. The many applications of such differential equations to discoveries in the physical sciences and engineering over the past three centuries has been one of the greatest intellectual success stories of all time.

Prior to the invention of calculus by Newton and Leibniz, most mathematics was of a discrete nature. It dealt with sets that had a finite number of elements and with infinite but "countable" sets such as the natural numbers. These sets often include continuous curves such as the conics which can be characterized by a small number of conditions. Discrete mathematics also deals with the solutions to algebraic equations, which are usually discrete sets. The twentieth century has witnessed the creation of many additional subjects in the discrete direction. This development was spurred on by the rapidly increasing use of mathematics in the social, behavioral, decisional, and system sciences, where the items under investigation are typically finite in number and not readily approximated by some continuous idealization. Moreover, nearly all aspects of the ongoing revolution in *digital* computers involve discrete considerations.

A metal bar on a minute level is composed of discrete molecules, atoms, and elementary particles. Many of its physical properties, however, can be determined by viewing the bar as a solid continuum and employing the analytical techniques of calculus. One applies the basic laws of physics to express the local (infinitesimal) properties of the bar in terms of differential equations. The solutions of these equations, in turn, provide an excellent description of the observed global behavior of the bar.

One of the greatest mathematical discoveries of the twentieth century in continuous mathematics is the famous fixed point theorem, published in 1912 by the Dutch mathematician L. E. J. Brouwer (1881–1966). It states that any continuous function f from a set S (with certain desirable properties) into the same set S has at least one *fixed point* x_0, that is $f(x_0) = x_0$. For example, for any continuous function $y = f(x)$ from the set $S = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ into S there is some number $x_0 \in S$ such that the curve $y = f(x)$ crosses the line $y = x$ at x_0. A head of hair must either have a seam (a parting of the hair, a discontinuity) or else a cowlick (where the hair stands straight up, a fixed point). Fixed points are realized in many physical and social phenomena such as equilibrium states in mechanics or stable prices in economics.



Most human heads have a fixed point, in the form of a whorl, sometimes called a cowlick, from which all the hair radiates. It would be impossible to cover a sphere with hair (or with radiating lines) without at least one such fixed point.

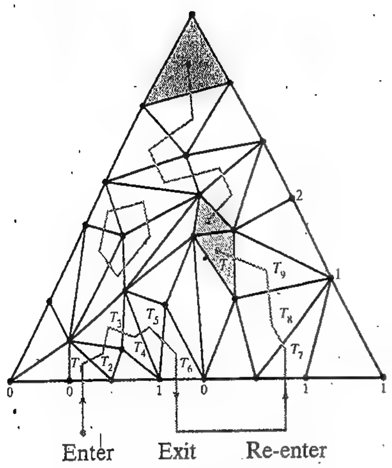


Figure 13.53 Arbitrary partitioning of a triangle

Like many of the outstanding results in continuous mathematics, Brouwer's theorem has an analogue in discrete mathematics, known as the labeling lemma of the German mathematician Emanuel Sperner, which appeared in 1928. Given a triangle with vertices 0, 1, and 2 (called $\Delta 012$) as shown in Figure 13.53, one can partition the interior of this triangle into (nonoverlapping) smaller triangles in any possible way.

One can then label each of the newly created vertices with any one of the numbers 0, 1, or 2. There is only one stipulation on how the new vertices on the perimeter of the triangle can be labeled: No vertex on a particular side of $\Delta 012$ can be labeled with the number appearing on the *main* vertex opposite (disjoint from) this side. For example, the side 01 of $\Delta 012$ cannot contain a vertex with the label 2. Sperner's lemma states that there must exist at least one elementary triangle inside $\Delta 012$ (or an *odd* number, in general) whose three vertices are "completely labeled" in the sense that they have all ~~three labels~~ 0, 1, and 2. There are three such triangles (shaded) in Figure 13.53.

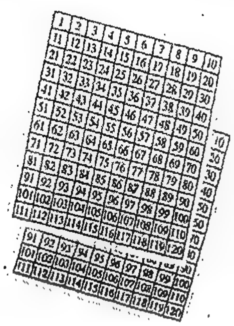
There is an elementary constructive proof of Sperner's lemma that uses he idea of a "path-following algorithm" published by Daniel I. A. Cohen in 1967. This proof is illustrated by the line shown in color (Figure 13.53). One can enter into $\Delta 012$ from outside via some elementary triangle T_1 whose perimeter edge carries the labels 0 and 1. The third vertex on this elementary triangle T_1 must be either 2 (in which case T_1 is a completely labeled elementary triangle), or else it is 0 or 1. In the latter cases, one can exit T_1 via a second 01 edge. One can continue to enter and exit subsequent elementary triangles $T_2, T_3, \ldots$ through successive 01 edges until one finally reaches an elementary triangle T_n that has completely labeled vertices 0, 1, and 2. (If this path were ever to exit $\Delta 012$, then there must be still another 01 edge on the perimeter of $\Delta 012$ since there is an odd number of 01 edges on the boundary of $\Delta 012$ where one can reenter the main triangle and continue along the path, as did occur once in Figure 13.53.)

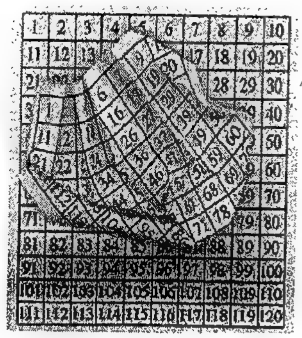
This path-following method also extends to a proof of Sperner's labeling lemma for dimensions higher than two. Furthermore, this approach is used in practical problems to arrive at a "very small" elementary triangle that can serve as an approximation to a fixed point, when it is excessively difficult or impossible to determine the precise location of the fixed point itself. In such cases, the broken path in Figure 13.53 can be viewed as the analogue in discrete mathematics of the continuous curves one arrives at when solving differential equations.

Around 1945, the great Hungarian-American mathematician John von Neumann (1903–1957) pointed out the need for approximating paths in a discrete manner when it proves too difficult to arrive at the exact continuous solutions to certain ordinary or partial differential equations. Cohen's constructive proof of Sperner's lemma, and work in the late 1960s by the mathematical economist Herbert Scarf on approximating equilibrium points, have paved the way for one very rich and extensive discrete theory that provides numerical approximations for continuous phenomena when the latter problems cannot be solved directly by the many analytic techniques of continuous mathematics.

Mathematical Essays

1. If your college offers a course on discrete mathematics, interview an instructor of the course, and using that interview as a basis, write an essay comparing continuous and discrete mathematics.
2. Draw several triangles and attempt to draw a counterexample for Sperner's labeling lemma. In each case, show how this lemma is satisfied.
3. Consider two sheets of paper containing the numbers 1 to 120, as shown in the photograph on the left.





If the top sheet is crumpled and dropped on the bottom sheet, the fixed point theorem tells us that one point must still be over its starting point. In the photograph on the right, it is a point in the region of the number 78. Perform this experiment several times in an attempt to find a counterexample. In each case, show how the fixed point theorem is satisfied.

4. **HISTORICAL QUEST** William Rowan Hamilton has been called the most renowned Irish mathematician. He was a child prodigy who read Greek, Hebrew, and Latin by the time he was five, and by the age of ten he knew over a dozen languages. Many mathematical advances are credited to Hamilton. For example, he developed vector methods in analytic geometry and calculus, as well as a system of algebraic quantities called **quaternions**, which occupied his energies for the last 22 years of his life. Hamilton pursued the study of quaternions with an almost religious fervor, but by the early twentieth century, the notation and terminology of vectors dominated. Much of the credit for the eventual emergence of vector methods goes not only to Hamilton, but also to the scientists James Clerk Maxwell (1831–1879), J. Willard Gibbs (1839–1903), and Oliver Heaviside (1850–1925). ■

WILLIAM ROWAN HAMILTON
1805–1865

   For this Quest, write a paper on quaternions.

5. Write a 500-word essay on the history of Green's theorem, Stokes' theorem, and the divergence theorem.
6. Write a report on four-dimensional geometry.
7. **Book Report** "We often hear that mathematics consists mainly of 'proving theorems.' Is a writer's job mainly that of 'writing sentences'? A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure that our minds are not playing tricks. Few people, if any, had dared write this out loud before Davis and Hersh. Theorems are not to mathematics what successful courses are to a meal. The nutritional analogy is misleading. To master mathematics is to master an intangible view." This quotation comes from the introduction to the book *The Mathematical Experience* by Philip J. Davis and Reuben Hersh (Boston: Houghton Mifflin, 1981). Read this book and prepare a book report.

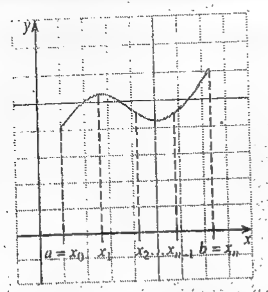
CHAPTERS 11–13 Cumulative Review

TABLE 13.1 Comparison of important integral theorems

Riemann Integral (Section 5.3)

$$\int_a^b f(x)\,dx$$

Subdivisions on the x-axis



Fundamental theorem of calculus (Section 5.4)

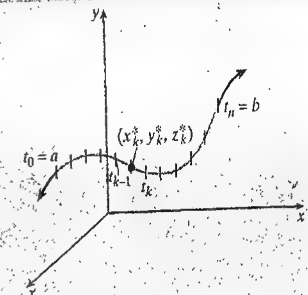
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

F is an antiderivative of f

Line Integral (Section 13.2)

$$\int_C f(x, y, z)\,ds$$

Subdivisions on a curve C in space



Fundamental theorem for line integrals (Section 13.3)

$$\int_C \mathbf{F}\cdot d\mathbf{R} = f(Q) - f(P)$$

If $\mathbf{F}$ is conservative with scalar potential f; that is, $\nabla f = \mathbf{F}$

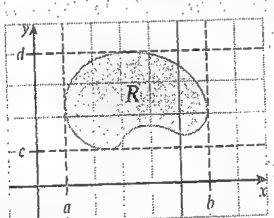
Green's theorem (Section 13.4): $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is continuously differentiable on the simply connected region D with positively oriented piecewise smooth boundary curve C. Then

$$\oint_C (M\,dx + N\,dy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Alternate forms include: $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_D (\operatorname{curl}\mathbf{F}\cdot\mathbf{k})\,dA$; $\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \iint_D \operatorname{div}\mathbf{F}\,dA$

Double Integral (Section 12.1)

$$\iint_R f(x, y)\,dA$$

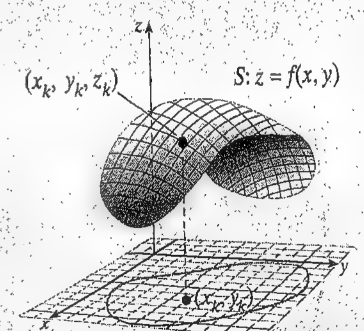


Partition of R into mn cells in the xy-plane

Surface Integral (Section 13.5)

$$\iint_S g(x, y, z)\,dS = \iint_R g(x, y, z)\sqrt{f_x^2 + f_y^2 + 1}\,dA$$

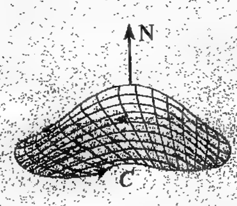
where $z = f(x, y)$



Partition of the surface S into n subregions

Stokes' theorem (Section 13.6)

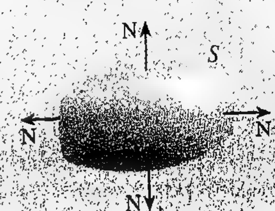
$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{N})\,dS$$



where C is the positively oriented boundary curve of the surface S with outward unit normal field N

The divergence theorem (Section 13.7; also known as Gauss' theorem)

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_D \operatorname{div}\mathbf{F}\,dV$$



where D is the solid region with closed boundary surface S.

Cumulative Review Problems for Chapters 11–13

1. **WHAT DOES THIS SAY?** Suppose you tell a fellow college student that you are about to finish a calculus course. The student has not had any mathematics beyond high school and asks you, "What is calculus?" Answer this question using your own words.

2. **WHAT DOES THIS SAY?** There are three great fundamental ideas in calculus: the limit, the derivative, and the integral. In your own words, explain each of these concepts.

3. **WHAT DOES THIS SAY?** Chapters 11–13 were concerned with functions of several variables. This is often referred to as **multivariable calculus**. In your own words, discuss what is meant by multivariable calculus.

Find f_x, f_y and f_{xy} for the functions whose equations are given in Problems 4–9.

4. $f(x, y) = 2x^2 + xy - 5y^3$

5. $f(x, y) = x^2 e^{y/x}$

6. $f(x, y) = \dfrac{x^2 - y^2}{x - y}$

7. $f(x, y) = y \sin^2 x + \cos xy$

8. $f(x, y) = e^{x+y}$

9. $f(x, y) = \dfrac{x^2 + y^2}{x - y}$

Evaluate the integrals in Problems 10–17.

10. $\displaystyle\int_0^1 \int_x^{2x} e^{y-x}\, dy\, dx$

11. $\displaystyle\int_0^4 \int_0^{\sqrt{x}} 3x^5\, dy\, dx$

12. $\displaystyle\int_0^1 \int_0^z \int_y^{y-z} (x + y + z)\, dx\, dy\, dz$

13. $\displaystyle\int_0^{15\pi} \int_0^{\pi} \int_0^{\sin\phi} \rho^3 \sin\phi\, d\rho\, d\theta\, d\phi$

14. $\displaystyle\iint_R e^{x+y}\, dA$; $R: 0 \le x \le 1; 0 \le y \le 1$

15. $\displaystyle\iint_R ye^{xy}\, dA$; $R: 0 \le x \le 1; 0 \le y \le 2$

16. $\displaystyle\iint_R \sin(x + y)\, dA$; $R: 0 \le x \le \frac{\pi}{2}; 0 \le y \le \frac{\pi}{4}$

17. $\displaystyle\iint_R x \sin xy\, dA$; $R: 0 \le x \le \pi; 0 \le y \le 1$

Evaluate the line integrals in Problems 18–21.

18. $\displaystyle\int_C (y^2 z\, dx + 2xyz\, dy + xy^2\, dz)$, where C is any path from $(0, 0, 0)$ to $(1, 1, 1)$

19. $\displaystyle\int_C (5xy\, dx + 10yz\, dy + z\, dz)$, where C is given by $x = t^2$, $y = t$, $z = 2t^3$ for $0 \le t \le 1$

20. , where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{k}$, and C is the boundary of the triangle $(1, 1, 1)$, $(1, 0, 1)$, $(0, 0, 1)$, traversed once counterclockwise as viewed from the origin

21. , where C is given by $\mathbf{R}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}$, for $0 \le t \le 2\pi$

22. Find the equations for the tangent plane and the normal line to $z = x^2 + y^2 + \sin xy$ at $P_0 = (0, 2, 4)$. The graph of this surface is shown in Figure 13.54.

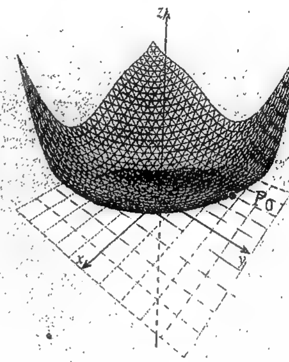


Figure 13.54 Graph of $z = x^2 + y^2 + \sin xy$

23. What are the dimensions of the closed rectangular box of fixed volume V_0 that has minimum surface area?

24. A manufacturer is planning to sell a new product at the price of \$350 per unit and estimates that if x thousand dollars is spent on development and y thousand dollars is spent on promotion, consumers will buy approximately

$$\frac{250y}{y + 2} + \frac{100x}{x + 5}$$

units of the product. If manufacturing costs for this product are \$150 per unit, how much should the manufacturer spend on development and how much on promotion to generate the largest possible profit, given the following circumstances?

a. Unlimited funds are available

b. The manufacturer has only \$11,000 to spend on development and promotion of the new product

25. A heat-seeking missile moves in a portion of space where the temperature (in degrees Celsius) at the point (x, y, z) is given by

$$T(x, y, z) = \tfrac{1}{10}(x^2 + y + z^3)$$

(x, y, and z are measured in kilometers).

a. Find the rate at which the temperature is changing as (x, y, z) moves from the point $P_0(-2, 9, 1)$ toward $Q(1, -3, 5)$.

b. If the heat-seeking missile is at P_0, in what direction will it travel to maximize the rate of heat increase?

c. What is the maximal rate of increase (in degrees Celsius per kilometer)?

26. Find the volume of the solid bounded above by the surface $z = x^2 + y^2 + 1$, below by the xy-plane, and on the sides by the cylinder $x^2 + y^2 = 1$.

27. Find the surface area of that portion of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 16$.

28. Find the center of mass of the lamina that covers the region R inside the circle $x^2 + y^2 = 4$ in the first quadrant, given that the density at any point (x, y) is $\delta\ (x, y) = x + y$.

29. A force field $\mathbf{F}(x, y) = (x - 2y)\mathbf{i} + (y - 2x)\mathbf{j}$ acts on an object moving in the plane. Show that $\mathbf{F}$ is conservative, and find a scalar potential for $\mathbf{F}$. How much work is done as the object moves from $(1, 0)$ to $(0, 1)$ along any path connecting these points?

30. A particle of weight w is acted on only by the constant gravitational force $\mathbf{F} = -w\mathbf{k}$. How much work is done in moving the weight along the helical path given by $x = \cos t$, $y = \sin t$, $z = t$ for $0 \le t \le 2\pi$?

14

Introduction to Differential Equations

CONTENTS

PREVIEW

We introduced and examined separable differential equations in Section 5.5 and first-order linear equations in Section 7.6. We examined applications such as orthogonal trajectories, flow of a fluid through an orifice, escape velocity of a projectile, carbon dating, the diastolic phase of blood pressure, learning curves, population models, dilution problems, and the flow of current in an *RL* circuit. In this chapter, we will extend our study of first-order differential equations by examining *homogeneous* and *exact* equations and then will investigate *second-order differential equations.*

PERSPECTIVE

The study of differential equations is such an extensive topic that even a brief survey of its methods and applications usually occupies a full course. Our goal in this chapter is to preview such a course by introducing some useful techniques for solving differential equations and by examining a few important applications.

14.1 First-Order Differential Equations

IN THIS SECTION review of separable differential equations, homogeneous differential equations, review of first-order linear differential equations, exact differential equations, Euler's method

We have discussed several different kinds of first-order differential equations so far in this text. In this section, we review our previous methods and introduce two new forms, *homogeneous* and *exact* differential equations.

REVIEW OF SEPARABLE DIFFERENTIAL EQUATIONS

Recall that a differential equation is just an equation involving derivatives or differentials. In particular, an ***n*th-order differential equation** in the dependent variable y with respect to the independent variable x is an equation in which the highest derivative of y that appears is d^ny/dx^n. A **general solution** of a differential equation is an expression that completely characterizes all possible solutions of the equation, and a **particular solution** is obtained by assigning specific values to the constants that appear in the general solution. An **initial value problem** involves solving a given differential equation subject to one or more initial conditions, such as $y(x_0) = y_0, y'(x_0) = y_1, \ldots$.

In Section 5.5, we defined a **separable differential equation** as one that can be written in the form

$$\frac{dy}{dx} = \frac{g(x)}{f(y)}$$

and observed that such an equation can be solved by separating the variables and integrating each side; that is,

$$\int f(y)\,dy = \int g(x)\,dx$$

EXAMPLE 1 Separable differential equation

Find the general solution of the differential equation

$$\frac{dy}{dx} = e^{-y}\sin x$$

Solution

Separate the variables and integrate:

$$\begin{aligned} \frac{dy}{dx} &= e^{-y}\sin x \\ e^y\,dy &= \sin x\,dx \\ \int e^y\,dy &= \int \sin x\,dx \\ e^y &= -\cos x + C \qquad \textit{Combine constants of integration.} \end{aligned}$$

This can also be written as $y = \ln|C - \cos x|$. ■

From the standpoint of applications, one of the most important separable differential equations is

$$\frac{dy}{dx} = ky$$

which occurs in the study of exponential growth and decay. You might review Table 7.3, which gives the solution for uninhibited growth or decay, logistic (or inhibited) growth, and limited growth functions. In this section, we consider an application of separable differential equations from chemistry.

EXAMPLE 2 Chemical conversion

Experiments in chemistry indicate that under certain conditions, two substances A and B will convert into a third substance C in such a way that the rate of conversion with respect to time is jointly proportional to the unconverted amounts of A and B. For simplicity, assume that one unit of C is formed from the combination of one unit of A and one unit of B, and assume that initially there are α units of A, β units of B, and no units of C present. Set up and solve a differential equation for the amount $Q(t)$ of C present at time t, assuming $\alpha \neq \beta$.

Solution

Since each unit of C is formed from one unit of A and one unit of B, it follows that at time t, $\alpha - Q(t)$ units of A and $\beta - Q(t)$ units of B remain unconverted. The specific rate condition can be expressed mathematically as

$$\frac{dQ}{dt} = k(\alpha - Q)(\beta - Q)$$

where k is a constant ($k > 0$ because $Q(t)$ is increasing).

To solve this equation, we separate the variables and integrate.

$$\int \frac{dQ}{(\alpha - Q)(\beta - Q)} = \int k\,dt$$

$$\int \frac{1}{\alpha - \beta}\left[\frac{-1}{\alpha - Q} + \frac{1}{\beta - Q}\right] dQ = \int k\,dt \qquad \text{Partial fraction decomposition}$$

$$\frac{1}{\alpha - \beta}[\ln(\alpha - Q) - \ln(\beta - Q)] = kt + C_1$$

$$\ln\left|\frac{\alpha - Q}{\beta - Q}\right| = (\alpha - \beta)kt + C_2$$

$$\frac{\alpha - Q}{\beta - Q} = Me^{(\alpha-\beta)kt} \qquad \text{Where } M = e^{C_2}$$

$$\alpha - Q = \beta Me^{(\alpha-\beta)kt} - QMe^{(\alpha-\beta)kt}$$

$$QMe^{(\alpha-\beta)kt} - Q = \beta Me^{(\alpha-\beta)kt} - \alpha$$

$$Q = \frac{\beta Me^{(\alpha-\beta)kt} - \alpha}{Me^{(\alpha-\beta)kt} - 1}$$

The initial condition tells us that $Q(0) = 0$, so that

$$0 = \frac{\beta Me^{0} - \alpha}{Me^{0} - 1}$$

$$0 = \beta M - \alpha$$

$$M = \frac{\alpha}{\beta}$$

Thus, $Q(t) = \dfrac{\beta\frac{\alpha}{\beta}e^{(\alpha-\beta)kt} - \alpha}{\frac{\alpha}{\beta}e^{(\alpha-\beta)kt} - 1} = \dfrac{\alpha\beta[e^{(\alpha-\beta)kt} - 1]}{\alpha e^{(\alpha-\beta)kt} - \beta}$. ■

HOMOGENEOUS DIFFERENTIAL EQUATIONS

Sometimes a first-order differential equation that is not separable can be put into separable form by a change of variables. A differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called a **homogeneous differential equation** if it can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

In other words, dy/dx is isolated on one side of the equation and the other side can be expressed as a function of y/x. We can then solve the differential equation by substitution.

To see how to solve such an equation, set $v = y/x$, so that

$$\begin{aligned}
vx &= y \\
\frac{d}{dx}(vx) &= \frac{d}{dx}(y) && \text{Take the derivative of both sides.} \\
v + x\frac{dv}{dx} &= \frac{dy}{dx} && \text{Product rule} \\
v + x\frac{dv}{dx} &= f(v) && \text{Substitution, } \frac{dy}{dx} = f\left(\frac{y}{x}\right) = f(v) \\
x\frac{dv}{dx} &= f(v) - v \\
\frac{dv}{f(v) - v} &= \frac{dx}{x}
\end{aligned}$$

The equation can now be solved by integrating both sides; remember to express your answer in terms of the original variables x and y (use $v = y/x$).

EXAMPLE 3 Homogeneous differential equation

Find the general solution of the equation $2xy\,dx + (x^2 + y^2)\,dy = 0$.

Solution

First, show that the equation is homogeneous by writing it in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$:

$$\begin{aligned}
2xy\,dx + (x^2 + y^2)\,dy &= 0 \\
\frac{dy}{dx} &= \frac{-2xy}{x^2 + y^2} \\
&= \frac{-2\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}
\end{aligned}$$

Let $v = \dfrac{y}{x}$ and $f(v) = \dfrac{-2v}{1 + v^2}$.

$$\frac{dv}{f(v)-v} = \frac{dx}{x}$$

$$\frac{dv}{\frac{-2v}{1+v^2} - v} = \frac{dx}{x}$$

$$-\int \frac{(1+v^2)\,dv}{v^3+3v} = \int x^{-1}\,dx$$

$$\int \left[\frac{\frac{1}{3}}{v} + \frac{\frac{2}{3}v}{v^2+3}\right] dv = -\int x^{-1}\,dx \qquad \text{Partial fraction decomposition}$$

$$\tfrac{1}{3}\ln|v| + \tfrac{2}{3}\left[\tfrac{1}{2}\ln|v^2+3|\right] = -\ln|x| + C_1$$

$$\tfrac{1}{3}\ln\left|v(v^2+3)\right| + \ln|x| = C_1$$

$$\ln\left|\frac{y}{x}\left[\left(\frac{y}{x}\right)^2+3\right]\right| + \ln|x^3| = C_2 \qquad \text{Substituting } v = \frac{y}{x};\ C_2 = 3C_1$$

$$\ln\left|\frac{y^3+3x^3}{x^3}\cdot x^3\right| = C_2$$

$$y^3 + 3x^3 = C \qquad \text{Where } C = e^{C_2}$$

This is the general solution of the given differential equation. ■

REVIEW OF FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

In Section 7.6, we considered differential equations of the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

WARNING Note that the coefficient of dy/dx is 1. If it is not, then divide by that nonzero coefficient.

Such an equation is said to be **first-order linear**, and we showed that its general solution is given by

$$y = \frac{1}{I(x)}\left[\int I(x)q(x)\,dx + C\right]$$

where $I(x)$ is the *integrating factor*

$$I(x) = e^{\int p(x)\,dx}$$

EXAMPLE 4 First-order linear differential equation

Find the solution of the differential equation

$$\frac{dy}{dx} + y\tan x = \sec x$$

that passes through the point $(\pi, 2)$.

Solution

Comparing the given first-order linear differential equation to the general first-order form, we see

$$p(x) = \tan x \quad \text{and} \quad q(x) = \sec x$$

The integrating factor is

$$I(x) = e^{\int \tan x\,dx} = e^{-\ln|\cos x|} = e^{\ln|(\cos x)^{-1}|} = (\cos x)^{-1} = \sec x$$

and the general solution is

$$\begin{aligned} y &= \frac{1}{\sec x}\left[\int (\sec x)(\sec x)\,dx + C\right] \\ &= \cos x\,[\tan x + C] \\ &= \sin x + C\cos x \end{aligned}$$

The initial condition gives

$$\begin{aligned} 2 &= \sin \pi + C\cos \pi \\ 2 &= -C \end{aligned}$$

so

$$y = \sin x - 2\cos x$$ ■

First-order linear differential equations appear in a variety of applications. In Section 7.6, we showed how first-order linear equations may be used to model mixture (dilution) problems as well as problems involving the current in an RL circuit (one with only a resistor, an inductor, and an electromotive force). In this section, we consider the motion of a body that falls in a resisting medium.

EXAMPLE 5 Motion of a body falling in a resisting medium

Consider an object with mass m that is initially at rest and is dropped from a great height (for example, from an airplane). Suppose the body falls in a straight line and the only forces acting on it are the downward force of the earth's gravitational attraction and a resisting upward force due to air resistance in the atmosphere. Assume that the resisting force is proportional to the velocity v of the falling body. Find equations for the velocity and position of the body's motion. Assume the distance $s(t)$ is measured down from the drop point.

Solution

The downward force is the weight mg of the body and the upward force is $-kv$, where k is a positive constant (the negative sign indicates that the force is directed upward). According to Newton's second law, the sum of the forces acting on a body at any time equals the product ma, where a is the acceleration of the body, that is

$$\underbrace{ma}_{\text{Sum of forces on the body}} = \underbrace{mg}_{\text{Force due to gravity}} - \underbrace{kv}_{\text{Resisting force}}$$

$$m\frac{dv}{dt} = mg - kv \qquad \text{Since } a = \frac{dv}{dt}$$

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

This is a first-order linear differential equation, where $p(t) = \frac{k}{m}$ and $q(t) = g$. The integrating factor is

$$I(t) = e^{\int k/m\, dt} = e^{kt/m}$$

so that the solution is

$$v = \frac{1}{e^{kt/m}}\left[\int e^{kt/m}(g)\,dt + C\right] = e^{-kt/m}\left[\frac{ge^{kt/m}}{k/m} + C\right] = \frac{mg}{k} + Ce^{-kt/m}$$

Because $v = 0$ when $t = 0$ (the body is initially at rest), it follows that

$$0 = \frac{mg}{k} + Ce^0 = \frac{mg}{k} + C$$

Solving, we obtain $C = -\frac{mg}{k}$, so

$$v = \frac{mg}{k} + \left(-\frac{mg}{k}\right)e^{-kt/m}$$

Now, to find the position $s(t)$, we use the fact that $v(t) = \frac{ds}{dt}$:

$$\frac{ds}{dt} = \frac{mg}{k} - \frac{mg}{k}e^{-kt/m}$$

$$\int ds = \int\left[\frac{mg}{k} - \frac{mg}{k}e^{-kt/m}\right]dt$$

$$s(t) = \frac{mg}{k}t - \frac{mg}{k}\frac{e^{-kt/m}}{-k/m} + C$$

$$= \frac{mg}{k}t + \frac{m^2g}{k^2}e^{-kt/m} + C$$

Because $s(0) = 0$ (the position s is measured from the point where the object is dropped), we find that

$$0 = \frac{mg}{k}(0) + \frac{m^2g}{k^2}e^0 + C \quad \text{so that} \quad -\frac{m^2g}{k^2} = C$$

Thus, the position is

$$s(t) = \frac{mg}{k}t + \frac{m^2g}{k^2}\left(e^{-kt/m} - 1\right)$$ ■

In the problem set, you are asked to show that no matter what the initial velocity may be, the velocity reached by the object in the long run (as $t \to +\infty$) is mg/k.

EXACT DIFFERENTIAL EQUATIONS

Sometimes a first-order differential equation can be written in the general form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

where the left side is an exact differential, namely,

$$df = M(x, y)\,dx + N(x, y)\,dy$$

for some function f. In this case, the given differential equation is appropriately called **exact** and since $df = 0$, its general solution is given by $f(x, y) = C$.

But how can we tell whether a particular first-order equation is exact, and if it is, how can we find f? Since

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

for a total differential (see Section 11.4), we must have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x, y)\,dx + N(x, y)\,dy$$

In particular, suppose M and N are continuously differentiable, with

$$\frac{\partial f}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y)$$

This will be true if and only if f satisfies the *cross-derivative* test

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

and then the function $f(x, y)$ is found by partial integration, exactly as we found the potential function of a conservative vector field in Section 13.3. The procedure for identifying and then solving an exact differential equation is illustrated in Example 6.

EXAMPLE 6 Exact differential equation

Find the general solution for $(2xy^3 + 3y)\,dx + (3x^2y^2 + 3x)\,dy = 0$.

Solution

Let $M(x, y) = 2xy^3 + 3y$ and $N(x, y) = 3x^2y^2 + 3x$, and apply the cross-derivative test

$$\frac{\partial M}{\partial y} = 6xy^2 + 3 \quad \text{and} \quad \frac{\partial N}{\partial x} = 6xy^2 + 3$$

Because $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact. To obtain a general solution, we must find a function f such that

$$\frac{\partial f}{\partial x} = 2xy^3 + 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + 3x$$

To find f, we integrate the first partial on the left with respect to x:

$$f(x, y) = \int (2xy^3 + 3y)\,dx = x^2y^3 + 3xy + u(y)$$

where u is a function of y. Taking the partial derivative of f with respect to y and comparing the result with $\partial f/\partial y$, we obtain

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2y^3 + 3xy + u(y)] = 3x^2y^2 + 3x + u'(y)$$

so that

$$3x^2y^2 + 3x = 3x^2y^2 + 3x + u'(y)$$
$$0 = u'(y)$$

This implies that u is a constant. Taking $u = 0$, we have $f = x^2y^3 + 3xy$, and the general solution to the exact differential equation is

$$x^2y^3 + 3xy = C$$ ■

A summary of strategies for identifying and solving various kinds of first-order differential equations is displayed in Table 14.1.

TABLE 14.1 Summary of Strategies for Solving First-Order Differential Equations

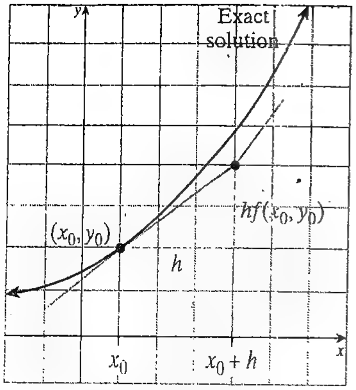
Form of Equation	Method	Solution
$\dfrac{dy}{dx} = \dfrac{g(x)}{f(y)}$	Separate the variables.	$\int f(y)\,dy = \int g(x)\,dx$
$\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$	Homogeneous—use a change of variable $v = y/x$.	$\int \dfrac{dv}{f(v) - v} = \int \dfrac{dx}{x}$
$\dfrac{dy}{dx} + p(x)y = q(x)$	Use the integrating factor $I(x) = e^{\int p(x)\,dx}$	$y = \dfrac{1}{I(x)}\left[\int I(x)q(x)\,dx + C\right]$
$M(x, y)\,dx + N(x, y)\,dy = 0$, where $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$	Exact—use partial integration to find f, where $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$	$f(x, y) = C$

EULER'S METHOD

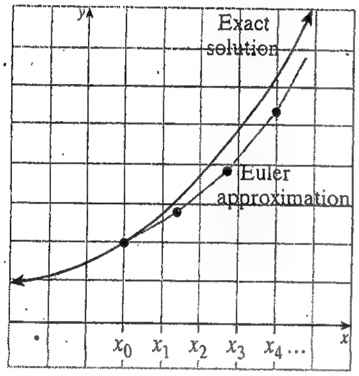
In Section 5.5, we introduced direction fields as a means for obtaining a "picture" of various solutions to a differential equation, but sometimes we need more than a rough graph of a solution. We now consider approximating a solution by numerical means. **Euler's method** is a simple procedure for obtaining a table of approximate values for the solution of a given initial value problem*

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

*In our discussion of Euler's method, we assume that the given initial value problem has a unique solution. It can be shown that such an initial value problem always has a unique solution if f and $\partial f/\partial y$ are both continuous in a neighborhood of (x_0, y_0). The proof of this result is beyond the scope of this text but can be found in most elementary differential equations texts.



a. The first Euler approximation



b. A sequence of Euler approximations

Figure 14.1 Graphical representation of Euler's method

The key idea in Euler's method is to increment x_0 by a small quantity h and then to estimate $y_1 = y(x_1)$ for $x_1 = x_0 + h$ by assuming x and y change by so little over the interval $[x_0, x_1]$ that $f(x, y)$ can be replaced by $f(x_0, y_0)$ for this interval. Solving the approximating initial value problem

$$\frac{dy}{dx} = f(x_0, y_0) \qquad y(x_0) = y_0$$

we obtain,

$$y - y_0 = f(x_0, y_0)(x - x_0)$$

In other words, we are approximating the solution curve $y = y(x)$ near (x_0, y_0) by the tangent line to the curve at this point, as shown in Figure 14.1a. We then repeat this process with (x_1, y_1) assuming the role of (x_0, y_0) to obtain an approximation of the solution $y = y(x)$ over the interval $x_1 \le x \le x_2$, where $x_2 = x_1 + h$ and

$$y_2 = y_1 + hf(x_1, y_1)$$

Continuing in this fashion, we obtain a sequence of line segments that approximates the shape of the solution curve as shown in Figure 14.1b. Euler's method is illustrated in the following example.

EXAMPLE 7 Euler's method

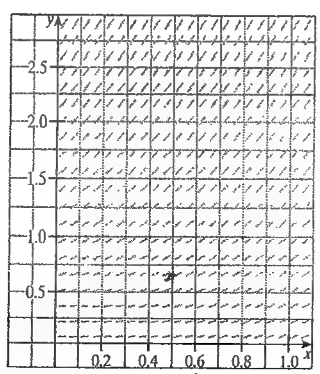
Use Euler's method with $h = 0.1$ to estimate the solution of the initial value problem

$$\frac{dy}{dx} = x + y^2 \qquad y(0) = 1$$

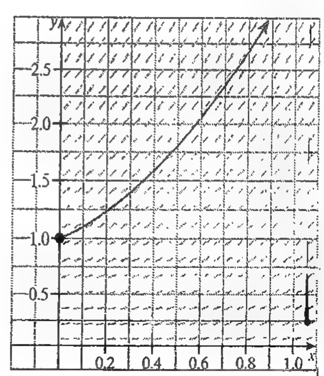
over the interval $0 \le x \le 0.5$.

Solution

Before using Euler's method, we might first look at a graphical solution. The slope field is shown in Figure 14.2a, and the particular solution through the point (0, 1) is shown in Figure 14.2b.



a. Slope field of $\frac{dy}{dx} = x + y^2$



b. Particular solution through (0, 1)

Figure 14.2 Graphical solution using a direction field

To use Euler's method for this example, we note

$$f(x, y) = x + y^2, \quad x_0 = 0, \quad y_0 = 1, \quad \text{and} \quad h = 0.1$$

We show the calculator (or computer) solution correct to four decimal places:

$$y_0 = y(0) = 1$$
$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1(0 + 1^2) = 1.1000$$
$$y_2 = y_1 + hf(x_1, y_1) = 1.1000 + 0.1(0.1 + 1.1000^2) = 1.2310$$
$$y_3 = y_2 + hf(x_2, y_2) = 1.2310 + 0.1(0.2 + 1.2310^2) \approx 1.4025$$
$$y_4 = y_3 + hf(x_3, y_3) = 1.4025 + 0.1(0.3 + 1.4025^2) \approx 1.6292$$
$$y_5 = y_4 + hf(x_4, y_4) = 1.6292 + 0.1(0.4 + 1.6292^2) \approx 1.9347$$

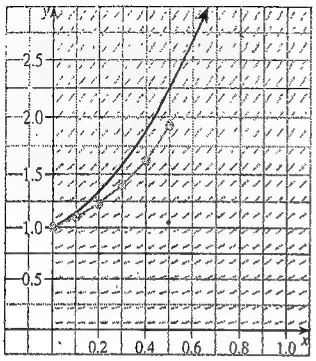


Figure 14.3 Solution by Euler's method

These points can be plotted to approximate the solution, as shown in Figure 14.3. Notice that we plotted these points by superimposing them on the direction field shown in Figure 14.2b. ■

Euler's method has educational value as the simplest numerical method for solving ordinary differential equations and can be found in most computer-assisted programs. However, as you might guess by looking at Figure 14.3, as you move away from (x_0, y_0), the error may accumulate. The Euler method can be improved in a variety of ways, most notably by a collection of procedures known as the *Runge–Kutta* and *predictor–corrector* methods. These methods are studied in more advanced courses.

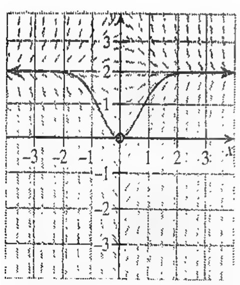
14.1 PROBLEM SET

A *Find the general solution of the differential equations in Problems 1–3 by separating variables.*

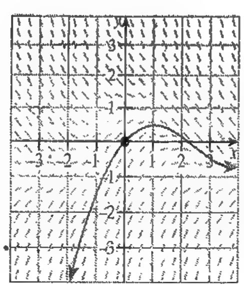
1. $xy\,dx = (x - 5)\,dy$
2. $(e^{2x} + 9)\dfrac{dy}{dx} = y$
3. $9\,dx - x\sqrt{x^2 - 9}\,dy = 0$

In Problems 4–7, solve the given first-order linear initial value problem. Compare your answer to the given graphical solution.

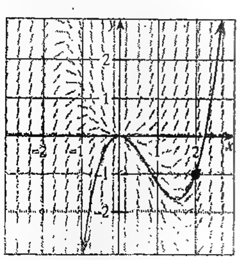
4. $\dfrac{dy}{dx} + 2xy = 4x,$
passing through (0, 0)



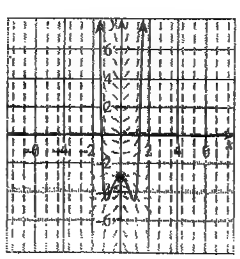
5. $\dfrac{dy}{dx} + y = \cos x,$
passing through (0, 0)



6. $x\dfrac{dy}{dx} - 2y = x^3,$
passing through (2, −1)



7. $\dfrac{dy}{dx} - 3xy = 5xe^{x^2},$
passing through (0, −3)



Show that each differential equation in Problems 8–10 is homogeneous and then find the general solution.

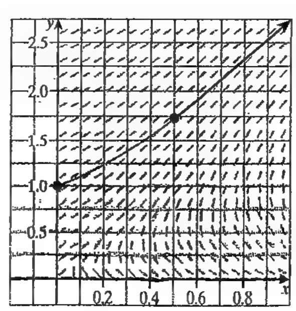
8. $(3x - y)\,dx + (x + 3y)\,dy = 0$
9. $xy\,dx - (2x^2 + y^2)\,dy = 0$
10. $(3x - y)\,dx + (x - 3y)\,dy = 0$

Show the differential equations in Problems 11–13 are exact and find the general solution.

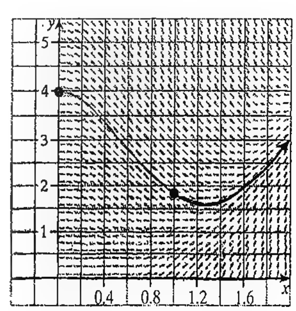
11. $(3x^2y + \tan y)\,dx + (x^3 + x\sec^2 y)\,dy = 0$
12. $(3x^2 - 10xy)\,dx + (2y - 5x^2 + 4)\,dy = 0$
13. $(2xy^3 + 3y - 3x^2)\,dx + (3x^2y^2 + 3x)\,dy = 0$

Estimate a solution for Problems 14–17 using Euler's method. For each of these problems, a direction field is given. Superimpose the segments from Euler's method on the given direction field.

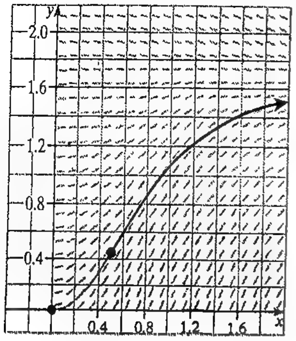
B 14. $\dfrac{dy}{dx} = \dfrac{x + y}{y - x},$
passing through (0, 1)
for $0 \le x \le 0.5$, $h = 0.1$



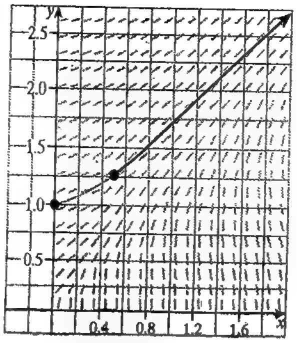
15. $\dfrac{dy}{dx} = 2x(x^2 - y),$
passing through (0, 4)
for $0 \le x \le 1$, $h = 0.2$



16. $\dfrac{dy}{dx} = \dfrac{5x - 3xy}{1 + x^2}$, passing through $(0, 0)$ for $0 \le x \le 0.5$, $h = 0.1$



17. $\dfrac{dy}{dx} = \dfrac{y^2 + 2x}{3y^2 - 2xy}$, passing through $(0, 1)$ for $0 \le x \le 0.5$, $h = 0.1$



*An **integrating factor** of the differential equation $M\,dx + N\,dy = 0$ is a function $g(x, y)$ such that*

$$g(x, y)M(x, y)\,dx + g(x, y)N(x, y)\,dy = 0$$

is exact. In Problems 18–19, find an integrating factor of the specified type for the given differential equations, then solve the equation.

18. $y\,dx + (y - x)\,dy = 0$; $g(x, y) = y^n$

19. $(x^2 + y^2)\,dx + (3xy)\,dy = 0$; $g(x, y) = x^n$

Identify each equation given in Problems 20–27 as separable, homogeneous, first-order linear, or exact, and then solve. A change of variable may be needed to put the given equation into one of these forms.

20. $(2xy^2 + 3x^2y - y^3)\,dx + (2x^2y + x^3 - 3xy^2)\,dy = 0$

21. $(1 + x)\,dy + \sqrt{1 - y^2}\,dx = 0$

22. $\left(\dfrac{2x}{y} - \dfrac{y^2}{x^2}\right) dx + \left(\dfrac{2y}{x} - \dfrac{x^2}{y^2} + 3\right) dy = 0$

23. $(x^2 - xy - x + y)\,dx - (xy - y^2)\,dy = 0$

24. $e^{y-x} \sin x\,dx - \csc x\,dy = 0$

25. $x^2 \dfrac{dy}{dx} + 2xy = \sin x$

26. $(3x^2 - y \sin xy)\,dx - (x \sin xy)\,dy = 0$

27. $(2x + \sin y - \cos y)\,dx + (x \cos y + x \sin y)\,dy = 0$

In Problems 28–31, solve the given initial value problem.

28. $\left(x \sin^2 \dfrac{y}{x} - y\right) dx + x\,dy = 0$; $x = \dfrac{4}{\pi}$, $y = 1$

29. $x\dfrac{dy}{dx} - 3y = x^3$; $x = 1$, $y = 1$

30. $[\sin(x^2 + y) + 2x^2 \cos(x^2 + y)]\,dx + [x \cos(x^2 + y)]\,dy = 0$; $x = 0$, $y = 0$

31. $y\dfrac{dy}{dx} = e^{x+2y} \sin x$; $x = 0$, $y = 0$

32. A population of foxes grows logistically until authorities decide to allow hunting at the constant rate of h foxes per month. The population $P(t)$ is then modeled by the differential equation

$$\frac{dP}{dt} = P(k - \ell P) - h$$

where $P(0) = P_0$.

a. Solve this equation in terms of k, ℓ, h, and P_0 for the case where $h < k^2/(4\ell)$.

b. What happens to $P(t)$ as $t \to \infty$?

33. A chemical in a solution diffuses from a compartment where the concentration is $C_0(t) = 7e^{-t}$ across a membrane with diffusion coefficient $k = 1.75$. The concentration $C(t)$ in the second compartment is modeled by

$$\frac{dC}{dt} = 1.75[C_0(t) - C(t)]$$

Solve this equation for $C(t)$, assuming that $C(0) = 0$.

34. The formula for the solution of a first-order linear equation requires the differential equation to be linear in x. Suppose, instead, that the equation is linear in y; that is, it can be written in the form

$$\frac{dx}{dy} + R(y)x = S(y)$$

a. Find a formula for the general solution of an equation that is first-order linear in y.

b. Use the formula obtained in part a to solve

$$y\,dx - 2x\,dy = y^4 e^{-y}\,dy$$

35. **Modeling Problem** A man is pulling a heavy sled along the ground by a rope of fixed length L. Assume the man begins walking at the origin of a coordinate plane and that the sled is initially at the point $(0, L)$. The man walks to the right (along the positive x-axis), dragging the sled behind him. When the man is at point M, the sled is at S as shown in Figure 14.4. Find a differential equation for the path of the sled and solve his differential equation.

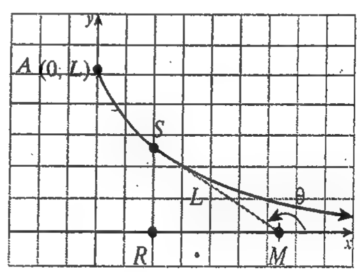


Figure 14.4 Sled problem

36. **Almost homogeneous equations** Sometimes a differential equation is not quite homogeneous but becomes homogeneous with a linear change of variable. Specifically, consider a differential equation of the form

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{rx + sy + t}\right)$$

a. Suppose $as \ne br$. Make the change of variable $x = X + A$ and $y = Y + B$, where A and B satisfy

$$\begin{cases} aA + bB + c = 0 \\ rA + sB + t = 0 \end{cases}$$

Show that with these choices for A and B the differential equation becomes homogeneous.

b. Apply the procedure outlined in part **a** to solve the differential equation

$$\frac{dy}{dx} = \left(\frac{-3x + y + 2}{x + 3y - 5}\right)$$

37. A **Riccati equation** is a differential equation of the form

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$$

Suppose we know that $y = u(x)$ is a solution of a given Riccati equation.

a. Change variables by setting

$$z = \frac{1}{y - u(x)}$$

Show that the given equation is transformed by this change of variables into the separable form

$$\frac{dz}{dx} + [2P(x)u(x) + Q(x)]z = -P(x)$$

b. Solve the first-order linear equation in z obtained in part **a** and explain how to find the general solution of the given Riccati equation.

c. Use the method outlined in parts **a** and **b** to solve the Riccati equation

$$\frac{dy}{dx} = \frac{1}{x^2}y^2 + \frac{2}{x}y - 2$$

Hint: There is a solution of the general form $y = Ax$.

38. Journal Problem *The American Mathematical Monthly**

Find all solutions of the Riccati equation

$$u' = u^2 + \frac{a}{x}u - b$$

$a, b \neq 0$, that are real rational functions of x.

*Problem E3055, Vol. 91, 1984, p. 515.

14.2 Second-Order Homogeneous Linear Differential Equations

IN THIS SECTION linear independence, solutions of the equation $ay'' + by' + cy = 0$, higher-order homogeneous linear equations, damped motion of a mass on a spring, reduction of order

LINEAR INDEPENDENCE

A **linear differential equation** is one of the general form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = R(x)$$

and if $a_n(x) \neq 0$, it is said to be of **order** n. It is *homogeneous* if $R(x) = 0$ and *nonhomogeneous* if $R(x) \neq 0$. In this section we focus attention on the homogeneous case, and we examine nonhomogeneous equations in the next section.

To characterize all solutions of the homogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$$

we require the following definition.

Linear Dependence and Independence

The functions $y_1, y_2, \ldots, y_n$ are said to be **linearly independent** if the equation

$$C_1y_1 + C_2y_2 + \cdots + C_ny_n = 0 \quad \text{for constants } C_1, C_2, \ldots, C_n$$

has only the trivial solution $C_1 = C_2 = \cdots = C_n = 0$ for all x in the interval I. Otherwise the y_k's are **linearly dependent**.

➡ **What This Says** A set of equations is *linearly independent* if none of the y_i can be written as a linear combination of the remaining y_j's $(j \neq i)$. Recall that for two functions to be equal, they must be equal at every point x.

The functions $y_1 = \cos x$ and $y_2 = x$ are linearly independent because the only way we can have $C_1 \cos x + C_2 x = 0$ for all x is for C_1 and C_2 both to be 0. However, $y_1 = 1$, $y_2 = \sin^2 x$, and $y_3 = \cos 2x$ are linearly dependent, because

$$C_1(1) + C_2(\sin^2 x) + C_3(\cos 2x) = 0 \qquad \text{for } C_1 = 1, C_2 = -2, C_3 = -1$$

It can be quite difficult to determine whether a given collection of functions y_1, $y_2, \ldots, y_n$ is linearly independent using the definition. An alternate approach to settling the issue of linear independence involves the following determinant function, which is named after Josef Hoëné de Wronski (see the Historical Quest in Problem 19).

Wronskian

The **Wronskian** $W(y_1, y_2, \ldots, y_n)$ of n functions $y_1, y_2, \ldots, y_n$ having $n-1$ derivatives on an interval I is defined to be the determinant function

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

THEOREM 14.1 **Determining linear independence with the Wronskian**

Suppose the functions $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ in the nth order homogeneous linear differential equation

$$a_n(x)\, y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_0(x)\, y = 0$$

are all continuous on a closed interval $[c, d]$. Then solutions $y_1, y_2, \ldots, y_n$ of this differential equation are linearly independent if and only if the Wronskian is nonzero; that is,

$$W(y_1, y_2, \ldots, y_n) \neq 0$$

throughout the interval $[c, d]$.

Proof The proof can be found in most differential equations texts. ❑

EXAMPLE 1 Showing linear independence

The functions $y_1 = e^{-x}$, $y_2 = xe^{-x}$, and $y_3 = e^{3x}$ are solutions of a certain homogeneous linear differential equation with constant coefficients. Show that these solutions are linearly independent.

Solution

$$W(e^{-x}, xe^{-x}, e^{3x}) = \begin{vmatrix} e^{-x} & xe^{-x} & e^{3x} \\ -e^{-x} & (1-x)e^{-x} & 3e^{3x} \\ e^{-x} & (x-2)e^{-x} & 9e^{3x} \end{vmatrix}$$

$$\begin{aligned} &= e^{-x}[9e^{3x}(1-x)e^{-x} - 3e^{3x}(x-2)e^{-x}] \\ &\quad - xe^{-x}[-9e^{3x}e^{-x} - 3e^{-x}e^{3x}] \\ &\quad + e^{3x}[-e^{-x}(x-2)e^{-x} - e^{-x}(1-x)e^{-x}] \\ &= 16e^{x} \end{aligned}$$

Because $16e^x \neq 0$, the functions are linearly independent. ■

The general solution of an nth-order homogeneous linear differential equation with constant coefficients can be characterized in terms of n linearly independent solutions. Here is the theorem that applies to the second-order case.

THEOREM 14.2 Characterization theorem

Suppose y_1 and y_2 are linearly independent solutions of the differential equation $ay'' + by' + cy = 0$; that is, $W(y_1, y_2) \neq 0$. Then the general solution of the equation is

$$y = C_1y_1 + C_2y_2 \qquad \text{for arbitrary constants } C_1, C_2$$

Proof We will prove that if y_1 and y_2 are linearly independent solutions, then $y = C_1y_1 + C_2y_2$ is also a solution. However, the proof that all solutions are of this form is beyond the scope of this book. Suppose y_1 and y_2 are solutions, so that

$$\begin{aligned} ay_1''(x) + by_1'(x) + cy_1(x) &= 0 \\ ay_2''(x) + by_2'(x) + cy_2(x) &= 0 \end{aligned}$$

If $y = C_1y_1 + C_2y_2$, we have

$$\begin{aligned} ay'' + by' + cy &= a[C_1y_1'' + C_2y_2''] + b[C_1y_1' + C_2y_2'] + c[C_1y_1 + C_2y_2] \\ &= C_1[ay_1'' + by_1' + cy_1] + C_2[ay_2'' + by_2' + cy_2] \\ &= 0 + 0 = 0 \end{aligned}$$

Thus, $y = C_1y_1 + C_2y_2$ is also a solution. ❑

SOLUTIONS OF THE EQUATION $ay'' + by' + cy = 0$

Thanks to the characterization theorem, we now know that once we have two linearly independent solutions y_1, y_2 of the equation $ay'' + by' + cy = 0$, we have them all because the general solution can be characterized as $y = C_1y_1 + C_2y_2$. Therefore, the whole issue of how to represent the solution of a second-order homogeneous linear equation with constant coefficients depends on finding two linearly independent solutions.

Recall that the general solution of the first-order equation $y' + ay = 0$ is $y = Ce^{-ax}$. Therefore, it is not unreasonable to expect the second-order equation $ay'' + by' + cy = 0$ to have one or more solutions of the form $y = e^{rx}$. If $y = e^{rx}$, then $y' = re^{rx}$ and $y'' = r^2e^{rx}$, and by substituting these derivatives into the equation $ay'' + by' + cy = 0$, we obtain

$$\begin{aligned} ay'' + by' + cy &= 0 \\ a(r^2e^{rx}) + b(re^{rx}) + ce^{rx} &= 0 \\ e^{rx}(ar^2 + br + c) &= 0 \\ ar^2 + br + c &= 0 \quad e^{rx} \neq 0 \end{aligned}$$

Thus, $y = e^{rx}$ is a solution of the given second-order differential equation if and only if $ar^2 + br + c = 0$. This equation is called the **characteristic equation** of $ay'' + by' + cy = 0$ and $ar^2 + br + c$ is the **characteristic polynomial**.

EXAMPLE 2 Characteristic equation with distinct real roots

Find the general solution of the differential equation $y'' + 2y' - 3y = 0$.

Solution

Begin by solving the characteristic equation:

$$r^2 + 2r - 3 = 0$$
$$(r-1)(r+3) = 0$$
$$r = 1, -3$$

The particular solutions are $y_1 = e^x$ and $y_2 = e^{-3x}$. Next, determine whether these equations are linearly independent by looking at the Wronskian.

$$W(e^x, e^{-3x}) = \begin{vmatrix} e^x & e^{-3x} \\ e^x & -3e^{-3x} \end{vmatrix} = e^x(-3e^{-3x}) - e^x e^{-3x} = -4e^{-2x}$$

Because $W(e^x, e^{-3x}) = -4e^{-2x} \neq 0$, the functions are linearly independent, and the characterization theorem tells us that the general solution is

$$y = C_1y_1 + C_2y_2 = C_1e^x + C_2\,e^{-3x}$$ ■

When a characteristic polynomial $r^2 + ar + b$ does not factor as easily as the one in Example 2, it may be necessary to find the roots r_1 and r_2 of the characteristic equation by applying the quadratic formula to obtain

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The *discriminant* for this equation, $b^2 - 4ac$, figures prominently in the following theorem.

THEOREM 14.3 Solution of $ay'' + by' + cy = 0$

If the characteristic equation $ar^2 + br + c = 0$ of the homogeneous linear differential equation $ay'' + by' + cy = 0$ has roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

then the general solution of the differential equation can be expressed in exactly one of the three following forms, depending on the sign of the discriminant $b^2 - 4ac$:

$b^2 - 4ac > 0$: The roots r_1 and r_2 are real and distinct and the general solution is

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

for arbitrary constants C_1 and C_2.

$b^2 - 4ac = 0$: The roots r_1 and r_2 are real and equal, $r_1 = r_2 = -\dfrac{b}{2a}$. The general solution is

$$y = C_1e^{r_1x} + C_2xe^{r_1x} = (C_1 + C_2x)e^{r_1x}$$

$b^2 - 4ac < 0$: The roots r_1 and r_2 are complex conjugates,

$$r_1 = \alpha + \beta i \quad \text{and} \quad r_2 = \alpha - \beta i,$$

where $\alpha = -\dfrac{b}{2a}$, $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$, and $i = \sqrt{-1}$.

The general solution is

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

Proof We have just seen that the solutions of the differential equation of the form $y = e^{rx}$ correspond to the solutions of the characteristic equation $ar^2 + br + c = 0$. The quadratic formula characterizes the three cases according to the discriminant of this equation, namely, $b^2 - 4ac$. For each case, we must find two linearly independent solutions.

$b^2 - 4ac > 0$: Let $y_1 = e^{r_1 x}$, and $y_2 = e^{r_2 x}$. Then

$$W(e^{r_1 x}, e^{r_2 x}) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = r_2 e^{(r_1 + r_2)x} - r_1 e^{(r_1 + r_2)x} = (r_2 - r_1) e^{(r_1 + r_2)x}$$

Because $r_2 \neq r_1$ and $e^{(r_1 + r_2)x} > 0$, we see $W(e^{r_1 x}, e^{r_2 x}) \neq 0$, so the functions are linearly independent. The characterization theorem tells us that the general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$b^2 - 4ac = 0$: In this case, the characteristic equation has one repeated root, $r = r_1 = r_2 = -\dfrac{b}{2a}$. Thus, $y_1 = e^{rx}$ is one solution, and it can be shown that $y_2 = xe^{rx}$ is a second, linearly independent solution (see Problem 18), so the general solution is

$$y = C_1 e^{rx} + C_2 x e^{rx}$$

$b^2 - 4ac < 0$: We must show that

$$y_1 = e^{\alpha x} \cos \beta x \quad \text{and} \quad y_2 = e^{\alpha x} \sin \beta x$$

are linearly independent and satisfy the given differential equation. ❑

EXAMPLE 3 Characteristic equation with repeated roots

Find the general solution of the differential equation $y'' + 4y' + 4y = 0$.

Solution

Solve the characteristic equation

$$\begin{aligned} r^2 + 4r + 4 &= 0 \\ (r + 2)^2 &= 0 \\ r &= -2 \qquad \text{(multiplicity 2)} \end{aligned}$$

The roots are $r_1 = r_2 = -2$. Thus, $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$, so the general solution of the differential equation is

$$y = C_1e^{-2x} + C_2xe^{-2x}$$ ■

EXAMPLE 4 Characteristic equation with complex roots

Find the general solution of the differential equation $2y''+3y' + 5y = 0$.

Solution

Solve the characteristic equation:

$$\begin{aligned}2r^2 + 3r + 5 &= 0\\ r &= \frac{-3 \pm \sqrt{9-4(2)(5)}}{2(2)}\\ &= \frac{-3 \pm \sqrt{31}\,i}{4}\end{aligned}$$

The roots are

$$r_1 = -\frac{3}{4} + \frac{\sqrt{31}}{4}i, \quad r_2 = -\frac{3}{4} - \frac{\sqrt{31}}{4}i$$

Thus, the general solution is

$$y = e^{(-3/4)x}\left[C_1 \cos\frac{\sqrt{31}}{4}x + C_2 \sin\frac{\sqrt{31}}{4}x\right]$$ ■

EXAMPLE 5 Second-order initial value problem

Solve $4y'' + 12y' + 9y = 0$ subject to $y(0) = 3$ and $y'(0) = -2$.

Solution

Solve

$$\begin{aligned}4r^2 + 12r + 9 &= 0\\ (2r+3)^2 &= 0\\ r_1 = r_2 &= -\frac{3}{2}\end{aligned}$$

Thus, the general solution is

$$y = C_1e^{(-3/2)x} + C_2xe^{(-3/2)x}$$

Because $y(0) = 3$ we have

$$\begin{aligned}3 &= C_1e^0 + C_2(0)e^0\\ 3 &= C_1\end{aligned}$$

Since $y'(0) = -2$, we find y':

$$y' = -\tfrac{3}{2}C_1e^{(-3/2)x} - \tfrac{3}{2}C_2xe^{(-3/2)x} + C_2e^{(-3/2)x}$$
$$y'(0) = -\tfrac{3}{2}C_1e^0 - \tfrac{3}{2}C_2(0)e^0 + C_2e^0$$
$$-2 = -\tfrac{3}{2}C_1 + C_2$$
$$-2 = -\tfrac{3}{2}(3) + C_2 \qquad \text{Because } C_1 = 3$$
$$\tfrac{5}{2} = C_2$$

Thus, the particular solution is

$$y = 3e^{(-3/2)x} + \tfrac{5}{2}xe^{(-3/2)x}$$ ■

HIGHER-ORDER HOMOGENEOUS LINEAR EQUATIONS

Homogeneous linear differential equations of order 3 or more with constant coefficients can be handled in essentially the same way as the second-order equations we have analyzed. As in the second-order case, some of the roots of the characteristic equations may be real and distinct, some may be real and repeated, and some may occur in complex conjugate pairs. But now, roots of the characteristic equation may occur more than twice, and when this happens, the linearly independent solutions are obtained by multiplying by increasing powers of x. For example, if 2 is a root of multiplicity 4 in the characteristic equation, the corresponding linearly independent solutions are e^{2x}, xe^{2x}, x^2e^{2x}, and x^3e^{2x}. The procedure for obtaining the general solution of an nth order linear homogeneous equation with constant coefficients is illustrated in the next two examples.

EXAMPLE 6 Characteristic equation with repeated roots

Solve $y^{(4)} - 5y''' + 6y'' + 4y' - 8y = 0$.

Solution

Solve the characteristic equation:

$$r^4 - 5r^3 + 6r^2 + 4r - 8 = 0$$

Because this is 4th degree, we use synthetic division and the rational root theorem (or a calculator) to find the roots -1, 2, 2, and 2. The general solution is

$$y = C_1e^{-x} + C_2e^{2x} + C_3xe^{2x} + C_4x^2e^{2x}$$ ■

EXAMPLE 7 Characteristic equation with repeated roots (some not real)

Solve $y^{(7)} + 8y^{(5)} + 16y''' = 0$.

Solution

Solve the characteristic equation:

$$r^7 + 8r^5 + 16r^3 = 0$$
$$r^3(r^4 + 8r^2 + 16) = 0$$
$$r^3(r^2 + 4)^2 = 0$$
$$r = 0 \text{ (multiplicity 3)}, \pm 2i \text{ (multiplicity 2)}$$

WARNING ⟹ The general solution of an nth order linear homogeneous differential equation involves n arbitrary constants $C_1, C_2, \ldots, C_n$. Always check to make sure your solution has the correct number of constants.

The roots (showing multiplicity) are 0, 0, 0, $2i$, $2i$, $-2i$, $-2i$. The general solution is

$$y = C_1 + C_2x + C_3x^2 + C_4\cos 2x + C_5\sin 2x + C_6x\cos 2x + C_7x\sin 2x \qquad \blacksquare$$

DAMPED MOTION OF A MASS ON A SPRING

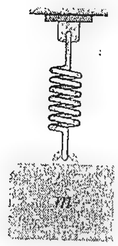


Figure 14.5 Oscillating spring

To illustrate an application of a second-order homogeneous linear differential equation, we will consider the motion of an oscillating spring, which we illustrate in Figure 14.5. Suppose an object suspended at the end of a spring is pulled down and then released. Hooke's law in physics says that a spring that is stretched or compressed x units from its natural length tends to restore itself to its natural length by a force whose magnitude F is proportional to x. Specifically, $F(x) = kx$, where the constant of proportionality k is called the **spring constant** and depends on the stiffness of the spring.

Suppose the mass of the spring is negligible compared to the mass m of the object on the spring. When the object is pulled down and released, the spring begins to oscillate, and its motion is determined by two forces, the weight mg of the object and the restoring force $F(x) = k_1x$ of the spring. According to Newton's second law of motion, the force acting on the object is ma, where $a = x''(t)$ is the acceleration of the object. If there are no other external forces acting on the object, the motion is said to be **undamped**, and the motion is governed by the second-order homogeneous equation.

$$mx''(t) = -k_1x(t) \qquad k_1 > 0$$
$$mx''(t) + k_1x(t) = 0$$

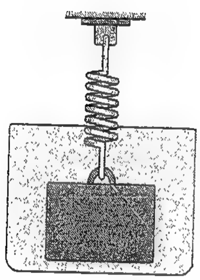


Figure 14.6 A damped spring system

Next, suppose the object is connected to a dashpot, or a device that imposes a damping force. A good example is a shock absorber in a car, which forces a spring to move through a fluid (see Figure 14.6). Experiments indicate that the shock absorber introduces a damping force proportional to the velocity $v = x'(t)$. Thus, the total force in this case is $-k_1x(t) - k_2x'(t)$, and Newton's second law tells us

$$mx''(t) = -k_1x(t) - k_2x'(t) \qquad k_1 > 0,\ k_2 > 0$$
$$mx''(t) + k_2x'(t) + k_1x(t) = 0$$

The constant k_2 is called the *damping constant* to contrast it from the spring constant k_1. The characteristic equation is

$$mr^2 + k_2r + k_1 = 0$$
$$r = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1m}}{2m}$$

The three cases that can occur correspond to different kinds of motion for the object on the spring.

overdamping: $k_2^2 - 4k_1m > 0$ In this case, both roots are real and negative, and the solution is of the form

$$x(t) = C_1e^{r_1t} + C_2e^{r_2t}$$

where

$$r_1 = -\frac{k_2}{2m} + \frac{1}{2m}\sqrt{k_2^2 - 4k_1 m} \quad \text{and}$$
$$r_2 = -\frac{k_2}{2m} - \frac{1}{2m}\sqrt{k_2^2 - 4k_1 m}$$

Note that the motion dies out eventually as $t \to +\infty$:

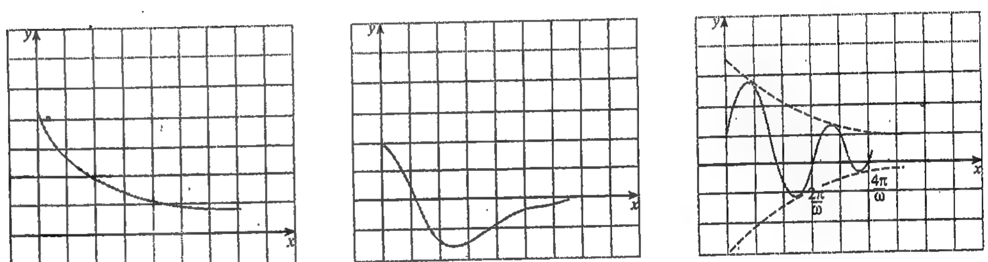
$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \left(C_1 e^{r_1 t} + C_2 e^{r_2 t}\right) = 0, \quad \text{because } r_1 < 0 \text{ and } r_2 < 0$$

Overdamping is illustrated in Figure 14.7a.

critical damping: $k_2^2 - 4k_1 m = 0$ The solution has the form

$$x(t) = (C_1 + C_2 t)e^{rt}$$

where $r_1 = r_2 = r = -\dfrac{k_2}{2m}$. In this case, the motion also eventually dies out (as $t \to +\infty$), because $r < 0$. This solution is shown in Figure 14.7b.



a. Overdamping $k_2^2 - 4k_1m > 0$ b. Critical damping $k_2^2 - 4k_1m = 0$ c. Underdamping $k_2^2 - 4k_1m < 0$

Figure 14.7 Damping motion

underdamping: $k_2^2 - 4k_1 m < 0$ In this case, the characteristic equation has complex roots, and the solution has the form

$$x(t) = e^{-k_2 t/(2m)}\left[C_1 \cos\left(\frac{t}{2m}\sqrt{4k_1 m - k_2^2}\right) + C_2 \sin\left(\frac{t}{2m}\sqrt{4k_1 m - k_2^2}\right)\right]$$

This can be written as

$$x(t) = Ae^{\alpha t}\cos(\omega t - C)$$

where

$$A = \sqrt{C_1^2 + C_2^2}; \quad \alpha = -\frac{k_2}{2m}; \quad \omega = \frac{1}{2m}\sqrt{4k_1 m - k_2^2}; \quad C = \tan^{-1}\frac{C_2}{C_1}$$

Because k_2 and m are both positive, α must be negative and we see that $x(t) \to 0$ as $t \to +\infty$. Notice that as the motion dies out, it oscillates with frequency $2\pi/\omega$, as shown in Figure 14.7c.

REDUCTION OF ORDER

Theorem 14.2 applies even when a, b, and c are functions of x instead of constants. In other words, the general solution of the equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

can be expressed as

$$y = C_1y_1 + C_2y_2$$

where $y_1(x)$ and $y_2(x)$ are any two linearly independent solutions. Sometimes, we can find one solution y_1 by observation and then obtain a second linearly independent solution y_2 by assuming that $y_2 = vy_1$, where $v(x)$ is a twice differentiable function of x. This procedure is called **reduction of order** because it involves solving the given second-order equation by solving two related first-order equations. The basic ideas of reduction of order are illustrated in the following example.

EXAMPLE 8 Reduction of order

Show that $y = x^2$ is a solution of the equation $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$ for $x > 0$, then find the general solution.

Solution

If $y_1 = x^2$, then $y_1' = 2x$ and $y_1'' = 2$. We have

$$y_1'' - \frac{3}{x}y_1' + \frac{4}{x^2}y_1 = 2 - \frac{3}{x}(2x) + \frac{4}{x^2}(x^2) = 0$$

so y_1 is a solution.

Next, let $y_2 = v(x)x^2$ and compute

$$y_2' = 2xv + x^2v' \quad \text{and} \quad y'' = 2v + 4xv' + x^2v''$$

Substitute these derivatives into the given equation:

$$(2v + 4xv' + x^2v'') - \frac{3}{x}(2xv + x^2v') + \frac{4}{x^2}(vx^2) = 0$$

$$x^2v'' + xv' = 0$$

$$\frac{v''}{v'} = \frac{-1}{x}$$

Integrate both sides of this equation:

$$\ln|v'| = -\ln x$$

$$v' = \frac{1}{x}$$

$$v = \ln x$$

Thus, $y_2 = vx^2 = x^2 \ln x$ is a second solution. To show that the solutions $y_1 = x^2$ and $y = x^2 \ln x$ are linearly independent, we compute the Wronskian:

$$W(x^2, x^2 \ln x) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & x + 2x \ln x \end{vmatrix} = x^3 \neq 0 \quad \text{Since } x > 0$$

It follows that the general solution of the given differential equation is

$$y = C_1x^2 + C_2x^2 \ln x$$ ■

14.2 PROBLEM SET

A *Find the general solution of the second-order homogeneous linear differential equations given in Problems 1–4.*

1. $y'' + y' = 0$
2. $y'' + y' - 2y = 0$
3. $3y'' + 11y' - 4y = 0$
4. $y'' - y = 0$

Find the general solution of the given higher-order homogeneous linear differential equations in Problems 5–7.

5. $y''' + 4y' = 0$
6. $y^{(4)} + y''' + 2y'' = 0$
7. $y^{(4)} + 2y''' + 2y'' + 2y' + y = 0$

Find the particular solution that satisfies the differential equations in Problems 8–10 subject to the specified initial conditions.

8. $y'' - 10y' + 25y = 0$; $y(0) = 1$, $y'(0) = -1$
9. $y'' - 12y' + 11y = 0$; $y(0) = 3$, $y'(0) = 11$
10. $y^{(4)} - y''' = 0$; $y(0) = 3$, $y'(0) = 0$, $y''(0) = 3$, $y'''(0) = 4$

In Problems 11–12, find the Wronskian, W, of the given set of functions and show that $W \neq 0$.

11. $\{e^{-2x}, e^{3x}\}$
12. $\{e^{-x}\cos x, e^{-x}\sin x\}$

B **13.** If $a^2 = b$, one solution of $y'' - 2ay' + by = 0$ is $y_1 = e^{ax}$. Use reduction of order to show that $y_2 = xe^{ax}$ is a second solution, and then show that y_1 and y_2 are linearly independent.

In Problems 14–17, a second-order differential equation and one solution $y_1(x)$ are given. Use reduction of order to find a second solution $y_2(x)$. Show that y_1 and y_2 are linearly independent and find the general solution.

14. $y'' + 6y' + 9y = 0$; $y_1 = e^{-3x}$
15. $xy'' + 4y' = 0$; $y_1 = 1$
16. $x^2y'' + 2xy' - 12y = 0$; $y_1 = x^3$
17. $(1 - x)^2y'' - (1 - x)y' - y = 0$; $y_1 = 1 - x$

18. Modeling Problem A 100-lb object is projected vertically upward from the surface of the earth with initial velocity 150 ft/s.

a. Modeling the object's motion with negligible air resistance, how long does it take for the object to return to earth?

b. Change the model to assume air resistance equal to half the object's velocity. Before making any computation, does your intuition tell you the object takes less or more time to return to earth this time than in part **a**? Now set up and solve a differential equation to actually determine the round-trip time. Were you right?

19. HISTORICAL QUEST Josef Hoëné (1778–1853) adopted the name Wronski when he was 32 years old, around the time he was married. Today, he is remembered for determinants now known as Wronskians, named by Thomas Muir (1844–1934) in 1882. Wronski's main work was in the philosophy of mathematics. For years his mathematical work, which contained many errors, was dismissed as unimportant, but in recent years closer study of his work revealed that he had some significant mathematical insight.

JOSEPH HOËNÉ DE WRONSKI 1778–1853

For this Historical Quest you are asked to write a paper on one of the great philosophical issues in the history of mathematics. Here are a few quotations to get you started:

Mathematics is discovered:

"... what is physical is subject to the laws of mathematics, and what is spiritual to the laws of God, and the laws of mathematics are but the expression of the thoughts of God."

—Thomas Hill, *The Uses of Mathesis; Bibliotheca Sacra.* p. 523.

"Our remote ancestors tried to interpret nature in terms of anthropomorphic concepts of their own creation and failed. The efforts of our nearer ancestors to interpret nature on engineering lines proved equally inadequate. Nature has refused to accommodate herself to either of these man-made molds. On the other hand, our efforts to interpret nature in terms of the concepts of pure mathematics have, so far, proved brilliantly successful ... from the intrinsic evidence of His creation, the Great Architect of the Universe now begins to appear as a pure mathematician."

—James H. Jeans, *The Mysterious Universe*, p. 142.

Mathematics is invented:

"There is an old Armenian saying, 'He who lacks sense of the past is condemned to live in the narrow darkness of his own generation.' Mathematics without history is mathematics stripped of its greatness: for, like the other arts-and mathematics is one of the supreme arts of civilization-it derives its grandeur from the fact of being a human creation."

—G. F. Simmons, *Differential Equations with Applications and Historical Notes*, Second Edition, McGraw-Hill, Inc., 1991, p. xix.

Discuss whether the significant ideas in mathematics are *discovered* or *invented*.

■

C **20.** The motion of a pendulum subject to frictional damping proportional to its velocity is modeled by the differential equation

$$mL\frac{d^2\theta}{dt^2} + kL\frac{d\theta}{dt} + mg\sin\theta = 0$$

where L is the length of the pendulum, m is the mass of the bob at its end, and θ is the angle the pendulum arm makes with the vertical (see Figure 14.8). Assume θ is small, so $\sin\theta$ is approximately equal to θ.

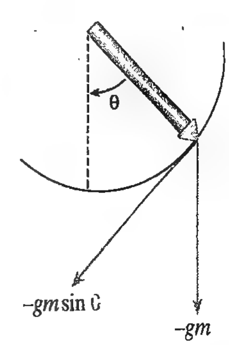


Figure 14.8 Damped pendulum

a. Solve the resulting differential equation for the case where $k^2 \geq 4gm^2/L$. What happens to $\theta(t)$ as $t \to \infty$?

b. If $k^2 < 4gm^2/L$, show that

$$\theta(t) = Ae^{-kt/(2m)} \cos\left(\sqrt{\frac{B}{L}}t + C\right)$$

for constants A, B, and C. What happens to $\theta(t)$ as $t \to \infty$ in this case?

c. For the situation in part **b**, show that the time difference between successive vertical positions is approximately

$$T = 2\pi m\sqrt{\frac{L}{4gm^2 - k^2L}}.$$

21. If there is no damping and there are no external forces, the motion of an object of mass m attached to a spring with spring constant k is governed by the differential equation $mx'' + kx = 0$. Show that the general solution of this equation is given by

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t + C\right), \qquad k > 0$$

where A and C are constants. This is called **simple harmonic motion** with frequency $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

22. Pendulum motion Suppose a ball of mass m is suspended at the end of a rod of length L and is set in motion swinging back and forth like a pendulum, as shown in Figure 14.9.

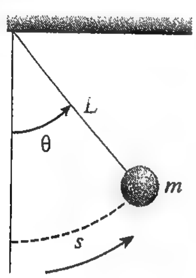


Figure 14.9 Pendulum motion

Let θ be the angle between the rod and the vertical at time t, so that the displacement of the ball from the equilibrium position is $s = L\theta$ and the acceleration of the ball's motion is

$$\frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2}.$$

a. Use Newton's second law to show that

$$mL\frac{d^2\theta}{dt^2} + mg\sin\theta = 0$$

Assume that air resistance and the mass of the rod are negligible.

b. When the displacement is small (θ close to 0), $\sin\theta$ may be replaced by θ. In this case, solve the resulting differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

How is the motion of the pendulum like the simple harmonic motion discussed in Problem 21?

23. Path of a projectile with variable mass A rocket starts from rest and moves vertically upward, along a straight line. Assume the rocket and its fuel initially weigh w kilograms and that the fuel initially weighs w_f kilograms. Further assume that the fuel is consumed at a constant rate of r kilograms per second (relative to the rocket). Finally, assume that gravitational attraction is the only external force acting on the rocket.

a. If $m(t)$ is the mass of the rocket and fuel at time t and $s(t)$ is the height above the ground at time t, it can be shown that

$$m(t)s''(t) + m'(t)v_0 + m(t)g = 0$$

where v_0 is the velocity of the exhaust gas in relation to the rocket. Express $m(t)$ in terms of w, r, v_0, and g, then integrate this differential equation to obtain the velocity $s'(t)$. Note that $s'(0) = 0$ because the rocket starts from rest.

b. Integrate the velocity $s'(t)$ to obtain $s(t)$.

c. At what time is all the fuel consumed?

d. How high is the rocket at the instant the fuel is consumed?

24. Journal Problem *Canadian Mathematical Bulletin** ■

Solve the differential equation

$$x^4y'' - (x^3 + 2axy)y' + 4ay^2 = 0$$

*Problem 331 by Murray S. Klamkin, Vol. 26, 1983, p. 126.

14.3 Second-Order Nonhomogeneous Linear Differential Equations

IN THIS SECTION nonhomogeneous equations, method of undetermined coefficients, variation of parameters, an application to *RLC* circuits

NONHOMOGENEOUS DIFFERENTIAL EQUATIONS

Next we will see how to solve a nonhomogeneous second-order linear equation of the general form $ay'' + by' + cy = F(x)$, where $F(x) \neq 0$. The key to our results is the following theorem.

THEOREM 14.4 **Characterization of the general solution of $ay'' + by' + cy = F(x)$**

Let y_p be a particular solution of the nonhomogeneous second-order linear equation $ay'' + by' + cy = F(x)$. Let y_h be the general solution of the related homogeneous equation $ay'' + by' + cy = 0$. Then the general solution of $ay'' + by' + cy = F(x)$ is given by the sum

$$y = y_h + y_p$$

Proof First, the sum $y = y_h + y_p$ is a solution of the nonhomogeneous equation $ay'' + by' + cy = F(x)$, because

$$\begin{aligned} ay'' + by' + cy &= a(y_h + y_p)'' + b(y_h + y_p)' + c(y_h + y_p) \\ &= ay_h'' + ay_p'' + by_h' + by_p' + cy_h + cy_p \\ &= (ay_h'' + by_h' + cy_h) + (ay_p'' + by_p' + cy_p) \\ &= 0 + F(x) \\ &= F(x) \end{aligned}$$

Conversely, if y is any solution of the nonhomogeneous equation, then $y - y_p$ is a solution of the related homogeneous equation because

$$\begin{aligned} a(y - y_p)'' + b(y - y_p)' + c(y - y_p) &= a(y'' - y_p'') + b(y' - y_p') + c(y - y_p) \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= F(x) - F(x) \\ &= 0 \end{aligned}$$

Thus, $y - y_p = y_h$ (because it is a solution of the homogeneous equation). Therefore, because y was *any* solution of the nonhomogeneous equation, it follows that $y = y_h + y_p$ is the general solution of the nonhomogeneous equation. ❑

➡ **What This Says** We can obtain the general solution of the nonhomogeneous equation $ay'' + by' + cy = F(x)$ by finding the general solution y_h of the related homogeneous equation $ay'' + by' + cy = 0$ and just one particular solution y_p of the given nonhomogeneous equation.

We can use the methods of the preceding section to find the general solution of the related homogeneous equation. We now develop two methods for finding particular solutions of the nonhomogeneous equation.

METHOD OF UNDETERMINED COEFFICIENTS

Sometimes it is possible to find a particular solution y_p of the nonhomogeneous equation $ay'' + by' + cy = F(x)$ by assuming a **trial solution** $\overline{y}_p$ of the same general form as $F(x)$. This procedure, called the **method of undetermined coefficients**, is illustrated in the following three examples, each of which has the related homogeneous equation $y'' + y' - 2y = 0$ with the general solution $y_h = C_1e^x + C_2e^{-2x}$.

EXAMPLE 1 Method of undetermined coefficients

Find $\overline{y}_p$ and the general solution for $y'' + y' - 2y = 2x^2 - 4x$.

Solution

The right side $F(x) = 2x^2 - 4x$ is a quadratic polynomial. Because derivatives of a polynomial are polynomials of lower degree, it seems reasonable to consider a trial solution that is also a polynomial of degree 2. That is, we "guess" that this equation has a particular solution $\overline{y}_p$ of the general form $\overline{y}_p = A_1x^2 + A_2x + A_3$. To find the constants A_1, A_2, and A_3, calculate

$$\overline{y}_p' = 2A_1x + A_2 \quad \text{and} \quad \overline{y}_p'' = 2A_1$$

Substitute the values for $\overline{y}_p$, $\overline{y}_p'$, and $\overline{y}_p''$ into the given equation:

$$\begin{aligned} y'' + y' - 2y &= 2x^2 - 4x \\ 2A_1 + 2A_1x + A_2 - 2(A_1x^2 + A_2x + A_3) &= 2x^2 - 4x \\ -2A_1x^2 + (2A_1 - 2A_2)x + (2A_1 + A_2 - 2A_3) &= 2x^2 - 4x \end{aligned}$$

Since this must be true for *every* x, we see that this is true only when the coefficients of each power of x on each of the equation match, so that

SMH See Problem 30 of Problem Set 3 of the *Student Mathematics Handbook*.

$$\begin{cases} -2A_1 = 2 & (x^2 \text{ terms}) \\ 2A_1 - 2A_2 = -4 & (x \text{ terms}) \\ 2A_1 + A_2 - 2A_3 = 0 & (\text{constant terms}) \end{cases}$$

Solve this system of equations simultaneously to find $A_1 = -1$, $A_2 = 1$, and $A_3 = -\frac{1}{2}$. Thus, a particular solution of the given nonhomogeneous equation is

$$y_p = A_1x^2 + A_2x + A_3 = -x^2 + x - \tfrac{1}{2}$$

and the general solution for the nonhomogeneous equation is

$$y = y_h + y_p = C_1e^x + C_2e^{-2x} - x^2 + x - \tfrac{1}{2}$$ ■

Comment: Notice that even though the constant term is zero in the polynomial function $F(x)$, we cannot assume that $y = A_1x^2 + A_2x$ is a suitable trial solution. In general, all terms of the same or lower degree that could possibly lead to the given right-side function $F(x)$ must be included in the trial solution.

EXAMPLE 2 Method of undetermined coefficients

Solve $y'' + y' - 2y = \sin x$.

Solution

Because the trial solution $\overline{y}_p$ is to be "like" the right-side function $F(x) = \sin x$, it seems that we should choose the trial solution to be $\overline{y}_p = A_1 \sin x$, but a sine function can have either a sine or a cosine in its derivatives, depending on how many derivatives are taken. Thus, to account for the $\sin x$ term, it is necessary to have both $\sin x$ and $\cos x$ in the trial solution, so we set

$$\overline{y}_p = A_1 \sin x + A_2 \cos x$$

Differentiating, we find

$$\overline{y}_p' = A_1 \cos x - A_2 \sin x \quad \text{and} \quad \overline{y}_p'' = -A_1 \sin x - A_2 \cos x$$

Substitute the values into the given equation:

$$\begin{aligned} y'' + y' - 2y &= \sin x \\ (-A_1 \sin x - A_2 \cos x) + (A_1 \cos x - A_2 \sin x) - 2(A_1 \sin x + A_2 \cos x) &= \sin x \\ (-3A_1 - A_2)\sin x + (A_1 - 3A_2)\cos x &= \sin x \end{aligned}$$

This gives the system

$$\begin{cases} -3A_1 - A_2 = 1 & (\sin x \text{ terms}) \\ A_1 - 3A_2 = 0 & (\cos x \text{ terms}) \end{cases}$$

with the solution $A_1 = -\frac{3}{10}$, $A_2 = -\frac{1}{10}$. Thus, the particular solution of the nonhomogeneous equation is $y_p = -\frac{3}{10}\sin x - \frac{1}{10}\cos x$, and the general solution is $y = y_h + y_p = C_1 e^x + C_2 e^{-2x} - \frac{3}{10}\sin x - \frac{1}{10}\cos x$. ■

Note: It can be shown that if y_1 is a solution of $ay'' + by' + cy = F(x)$ and y_2 is a solution of $ay'' + by' + cy = G(x)$, then $y_1 + y_2$ will be a solution of $ay'' + by' + cy = F(x) + G(x)$. (See Problem 24.) This is called the **principle of superposition.** For instance, by combining the results of Examples 1 and 2, we see that a particular solution of the nonhomogeneous linear equation

$$y'' + y' - 2y = 2x^2 - 4x + \sin x$$

is

$$y_p = \overbrace{-x^2 + x - \frac{1}{2}}^{\text{Solution for } F = 2x^2 - 4x} \underbrace{- \frac{3}{10}\sin x - \frac{1}{10}\cos x}_{\text{Solution for } G = \sin x}$$

EXAMPLE 3 Method of undetermined coefficients

Solve $y'' + y' - 2y = 4e^{-2x}$.

Solution

At first glance (looking at $F(x) = 4e^{-2x}$), it may seem that the trial solution should be $\overline{y}_p = Ae^{-2x}$. However, $y_1 = e^{-2x}$ is a solution of $y_1'' + y_1' - 2y_1 = 0$, so it cannot possibly also satisfy the given nonhomogeneous equation.

To deal with this situation, multiply the usual trial solution by x and consider the trial solution $\overline{y}_p = Axe^{-2x}$. Differentiating, we find

$$\overline{y}_p' = A(1 - 2x)e^{-2x} \quad \text{and} \quad \overline{y}_p'' = A(4x - 4)e^{-2x}$$

and by substituting into the given equation, we obtain

$$\begin{aligned} y'' + y' - 2y &= 4e^{-2x} \\ A(4x - 4)e^{-2x} + A(1 - 2x)e^{-2x} - 2Axe^{-2x} &= 4e^{-2x} \\ (4Ax - 4A + A - 2Ax - 2Ax)e^{-2x} &= 4e^{-2x} \\ -3Ae^{-2x} &= 4e^{-2x} \\ A &= -\tfrac{4}{3} \end{aligned}$$

Thus, $\bar{y}_p = -\frac{4}{3}xe^{-2x}$, so that the general solution is

$$y = y_h + y_p = C_1e^x + C_2e^{-2x} - \frac{4}{3}xe^{-2x}$$

The procedure illustrated in Examples 1–3 can be applied to a differential equation $y'' + ay' + by = F(x)$ only when $F(x)$ has one of the following forms:

a. $F(x) = P_n(x)$, a polynomial of degree n

b. $F(x) = P_n(x)e^{kx}$

c. $F(x) = e^{kx}\,[P_n(x)\cos\alpha x + Q_n(x)\sin\alpha x]$, where $Q_n(x)$ is another polyomial of degree n

We can now describe the **method of undetermined coefficients.**

Method of Undetermined Coefficients

To solve $ay'' + by' + cy = F(x)$ when $F(x)$ is one of the forms in the preceding list:

1. The solution is of the form $y = y_h + y_p$, where y_h is the general solution of the related homogeneous equation and y_p is a particular solution.
2. Find y_h by solving the related homogeneous equation

$$ay'' + by' + cy = 0$$

3. Find y_p by picking an appropriate trial solution $\bar{y}_p$:

Form of $F(x)$	Corresponding trial expression $\bar{y}_p$
a. $P_n(x) = c_nx^n + \cdots + c_1x + c_0$	$A_nx^n + \cdots + A_1x + A_0$
b. $P_n(x)e^{kx}$	$[A_nx^n + A_{n-1}x^{n-1} + \cdots + A_0]e^{kx}$
c. $e^{kx}\,[P_n(x)\cos\alpha x + Q_n(x)\sin\alpha x]$	$e^{kx}\,[(A_nx^n + \cdots + A_0)\cos\alpha x$ $+ B_nx^n + \cdots + B_0)\sin\alpha x]$

4. If no term in the trial expression $\bar{y}_p$ appears in the general homogeneous solution y_h, the particular solution can be found by substituting $\bar{y}_p$ into the equation $ay'' + by' + cy = F(x)$ and solving for the undetermined coefficients.
5. If any term in the trial expression $\bar{y}_p$ appears in y_h, multiply $\bar{y}_p$ by x^k, where k is the smallest integer such that no term in $x^k\bar{y}_p$ is a solution of $ay'' + by' + cy = 0$. Then proceed as in step 4, using $x^k\bar{y}_p$ as the trial solution.

EXAMPLE 4 Finding trial solutions

Determine a suitable trial solution for undetermined coefficients in each of the given cases.

a. $y'' - 4y' + 4y = 3x^2 + 4e^{-2x}$

b. $y'' - 4y' + 4y = 5xe^{2x}$

c. $y'' + 2y' + 5y = 3e^{-x}\cos 2x$

Solution

a. The related homogeneous equation $y'' - 4y' + 4y = 0$ has the characteristic equation $r^2 - 4r + 4 = 0$, which has the root 2 of multiplicity two. Thus, the general homogeneous solution is

$$y_h = C_1e^{2x} + C_2xe^{2x}$$

The part of the trial solution for the nonhomogeneous equation that corresponds to $3x^2$ is $A_0 + A_1x + A_2x^2$ and the part that corresponds to $4e^{-2x}$ is Be^{-2x}. Since neither part includes terms in y_h, we apply the principle of superposition to conclude that

$$\overline{y}_p = A_0 + A_1x + A_2x^2 + Be^{-2x}$$

b. We know from part **a** that the general homogeneous solution is

$$y_h = C_1e^{2x} + C_2xe^{2x}$$

The expected trial solution for $5xe^{2x}$ would be $(A_0 + A_1x)e^{2x}$, but this expression is contained in y_h. If we multiply by x, part of $(A_0+A_1x)xe^{2x}$ is still contained in y_h, so we multiply by x again to obtain

$$\overline{y}_p = (A_0 + A_1x)x^2e^{2x}$$

c. The related homogeneous equation $y'' + 2y' + 5y = 0$ has the characteristic equation $r^2 + 2r + 5 = 0$. This has complex conjugate roots $r = -1 \pm 2i$, so the general homogeneous solution is

$$y_h = e^{-x}[C_1 \cos 2x + C_2 \sin 2x]$$

Ordinarily, the trial solution for the nonhomogeneous equation would be of the form $e^{-x}[A \cos 2x + B \sin 2x]$, but part of this expression is in y_h. Therefore, we multiply by x to obtain the trial solution

$$\overline{y}_p = xe^{-x}[A \cos 2x + B \sin 2x]$$ ■

VARIATION OF PARAMETERS

The method of undetermined coefficients applies only when the coefficients a, b, and c are constant in the nonhomogeneous linear equation $ay'' + by' + cy = F(x)$ and the function $F(x)$ has the same general form as a solution of a second-order homogeneous linear differential equation with constant coefficients. Even though many important applications are modeled by differential equations of this type, there are other situations that require a more general procedure.

Our next goal is to examine a method of J. L. Lagrange (see Historical Quest, Problem 55 in Section 11.8) called **variation of parameters**, which can be used to find a particular solution of any nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = F(x)$$ *Note the leading coefficient is 1.*

where P, Q, and F are continuous.

To use variation of parameters assumes that we must be able to find two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the related homogeneous equation,

$$y'' + P(x)y' + Q(x)y = 0$$

In practice, if $P(x)$ and $Q(x)$ are not both constants, these may be difficult to find, but once we have them, we assume there is a solution of the nonhomogeneous equation of the form

$$y_p = uy_1 + vy_2$$

Differentiating this expression y_p, we obtain

$$y'_p = u'y_1 + v'y_2 + uy'_1 + vy'_2$$

To simplify, assume that

$$u'y_1 + v'y_2 = 0$$

Remember, we need to find only *one* particular solution y_p, and if imposing this side condition makes it easier to find such a y_p, so much the better! With the side condition, we have

$$y'_p = uy'_1 + vy'_2$$

and by differentiating again, we obtain

$$y''_p = uy''_1 + u'y'_1 + vy''_2 + v'y'_2$$

Next, we substitute our expressions for y'_p and y''_p into the given differential equation:

$$\begin{aligned} F(x) &= y''_p + P(x)y'_p + Q(x)y_p \\ &= (uy''_1 + u'y'_1 + vy''_2 + v'y'_2) + P(x)(uy'_1 + vy'_2) + Q(x)(uy_1 + vy_2) \end{aligned}$$

This can be rewritten as

$$u[y''_1 + P(x)y'_1 + Q(x)y_1] + v[y''_2 + P(x)y'_2 + Q(x)y_2] + u'y'_1 + v'y'_2 = F(x)$$

Because y_1 and y_2 are solutions of $y'' + P(x)y' + Q(x)y = 0$, we have

$$u\underbrace{[y''_1 + P(x)y'_1 + Q(x)y_1]}_{0} + v\underbrace{[y''_2 + P(x)y'_2 + Q(x)y_2]}_{0} + u'y'_1 + v'y'_2 = F(x)$$

or

$$u'y'_1 + v'y'_2 = F(x)$$

Thus, the parameters u and v must satisfy the system of equations

$$\begin{cases} u'y_1 + v'y_2 = 0 \\ u'y'_1 + v'y'_2 = F(x) \end{cases}$$

Solve this system to obtain

$$u' = \frac{-y_2F(x)}{y_1y'_2 - y_2y'_1} \quad \text{and} \quad v' = \frac{y_1F(x)}{y_1y'_2 - y_2y'_1}$$

where in each case the denominator is not zero, because it is the Wronskian of the linearly independent solutions y_1, y_2 of the related homogeneous differential equation. Integrating, we find

$$u(x) = \int \frac{-y_2F(x)}{y_1y'_2 - y_2y'_1}\,dx \quad \text{and} \quad v(x) = \int \frac{y_1F(x)}{y_1y'_2 - y_2y'_1}\,dx$$

and by substituting into the expression

$$y_p = uy_1 + vy_2$$

we obtain a particular solution of the given differential equation. Here is a summary of the procedure we have described.

WARNING The coefficient of y'' must be 1 for this method.

Variation of Parameters

To find the general solution of $y'' + P(x)y' + Q(x)\,y = F(x)$,

Step 1. Find the general solution, $y_h = C_1y_1 + C_2y_2$ to the related homogeneous equation, $y'' + P(x)y' + Q(x)y = 0$.

Step 2. Set $y_p = uy_1 + vy_2$ and substitute into the formulas:

$$u' = \frac{-y_2F(x)}{y_1y_2' - y_2y_1'} \qquad v' = \frac{y_1F(x)}{y_1y_2' - y_2y_1'}$$

Step 3. Integrate u' and v' to find u and v.

Step 4. A particular solution is $y_p = uy_1 + vy_2$, and the general solution is $y = y_h + y_p$.

These ideas are illustrated in the next example.

EXAMPLE 5 Variation of parameters

Solve $y'' + 4y = \tan 2x$.

Solution

Notice that this problem cannot be solved by the method of undetermined coefficients, because the right-side function $F(x) = \tan 2x$ is not one of the forms for which the procedure applies.

To apply variation of parameters, begin by solving the related homogeneous equation $y'' + 4y = 0$. The characteristic equation is $r^2 + 4 = 0$ with roots $r = \pm 2i$. These complex roots have $\alpha = 0$ and $\beta = 2$ so that the general solution is

$$y = e^0[C_1 \cos 2x + C_2 \sin 2x] = C_1 \cos 2x + C_2 \sin 2x$$

This means $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$. Set $y_p = uy_1 + vy_2$, where

$$u' = \frac{-y_2F(x)}{y_1y_2' - y_2y_1'} = \frac{-\sin 2x \tan 2x}{2\cos 2x \cos 2x + 2\sin 2x \sin 2x} = -\frac{\sin^2 2x}{2\cos 2x}$$

and

$$v' = \frac{y_1F(x)}{y_1y_2' - y_2y_1'} = \frac{\cos 2x \tan 2x}{2\cos 2x \cos 2x + 2\sin 2x \sin 2x} = \tfrac{1}{2}\sin 2x$$

Integrating, we obtain

$$u(x) = \int -\frac{\sin^2 2x}{2\cos 2x}\,dx \qquad\qquad v(x) = \int \tfrac{1}{2}\sin 2x\,dx$$

$$= -\tfrac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx \qquad\qquad = -\tfrac{1}{4}\cos 2x$$

$$= -\tfrac{1}{2}\int (\sec 2x - \cos 2x)\,dx$$

$$= -\tfrac{1}{2}\left[\tfrac{1}{2}\ln|\sec 2x + \tan 2x| - \frac{\sin 2x}{2}\right].$$

Thus, a particular solution is

$$\begin{aligned} y_p &= uy_1 + vy_2 \\ &= \left[-\tfrac{1}{4}\ln|\sec 2x + \tan 2x| + \tfrac{1}{4}\sin 2x\right]\cos 2x + \left(-\tfrac{1}{4}\cos 2x\right)\sin 2x \\ &= -\tfrac{1}{4}(\cos 2x)\ln|\sec 2x + \tan 2x| \end{aligned}$$

Finally, the general solution is

$$y = y_h + y_p = C_1\cos 2x + C_2\sin 2x - \tfrac{1}{4}(\cos 2x)\ln|\sec 2x + \tan 2x| \quad \blacksquare$$

AN APPLICATION TO *RLC* CIRCUITS

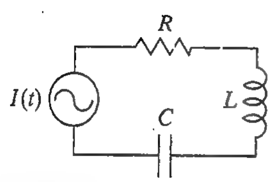


Figure 14.10 An *RLC* circuit

An important application of second-order linear differential equations is in the analysis of electric circuits. Consider a circuit with constant resistance R, inductance L, and capacitance C. Such a circuit is called an ***RLC* circuit** and is illustrated in Figure 14.10.

If $I(t)$ is the current in the circuit at time t and $Q(t)$ is the total charge on the capacitor, it is shown in physics that IR is the voltage drop across the resistance, Q/C is the voltage drop across the capacitor, and $L\,dI/dt$ is the voltage drop across the inductance. According to Kirchhoff's second law for circuits, the impressed voltage $E(t)$ in a circuit is the sum of the voltage drops, so that the current $I(t)$ in the circuit satisfies

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

By differentiating both sides of this equation and using the fact that $I = \dfrac{dQ}{dt}$, we can write

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$$

For instance, suppose the voltage input is sinusoidal—that is, $E(t) = A\sin\omega t$. Then we have $dE/dt = A\omega\cos\omega t$, and the second-order linear differential equation used in the analysis of the circuit is

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = A\omega\cos\omega t$$

To solve this equation, we proceed in the usual way, solving the related homogeneous equation and then finding a particular solution of the nonhomogeneous system. The solution to the related homogeneous equation is called the **transient current** because the current described by this solution usually does not last very long. The part of the nonhomogeneous solution that corresponds to transient current 0 is called the **steady-state current**. Several problems dealing with *RLC* circuits are outlined in the problem set.

14.3 PROBLEM SET

A *In Problems 1–5, find a trial solution $\overline{y}_p$ for use in the method of undetermined coefficients.*

1. $y'' - 6y' = e^{2x}$
2. $y'' + 2y' + 2y = e^{-x}$
3. $y'' + 2y' + 2y = e^{-x}\sin x$
4. $y'' + 4y' + 5y = e^{-2x}(x + \cos x)$
5. $y'' + 4y' + 5y = (e^{-x}\sin x)^2$

Use the method of undetermined coefficients to find the general solution of the nonhomogeneous differential equations given in Problems 6–11.

6. $y'' + 6y' + 5y = 2e^{x} - 3e^{-3x}$
7. $y'' + 2y' + 2y = \cos x$
8. $y'' - 6y' + 13y = e^{-3x}\sin 2x$
9. $y'' - y' = x^3 - x + 5$
10. $y'' - y' = (x - 1)e^{x}$
11. $y'' - y' - 6y = e^{-2x} + \sin x$

Use variation of parameters to solve the differential equations given in Problems 12–16.

12. $y'' + y = \tan x$
13. $y'' - y' - 6y = x^2e^{2x}$
14. $y'' + 3y' + 2y = \dfrac{e^x}{1 + e^x}$
15. $y'' - 4y' + 4y = \dfrac{e^{2x}}{1 + x}$
16. $y'' - y' = e^{-2x}\cos e^{-x}$

B *In each of Problems 17–20, use either undetermined coefficients or variation of parameters to find the particular solution of the differential equation that satisfies the specified initial conditions.*

17. $y'' - y' = 2\cos^2 x$; $y(0) = y'(0) = 0$
18. $y'' - y' = e^{-2x}\cos e^{-x}$; $y(0) = y'(0) = 0$
19. $y'' + 9y = 4e^{3x}$; $y(0) = 0, y'(0) = 2$
20. $y'' - 6y = x^2 - 3x$; $y(0) = 3, y'(0) = -1$
21. Find the general solution of the differential equation $y'' + y' - 6y = F(x)$, where F is the function whose graph is shown in Figure 14.11. *Hint:* Set up and solve two separate differential equations.

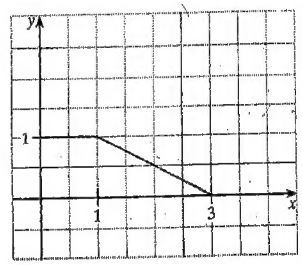


Figure 14.11 Graph of F for Problem 21

22. Find the steady-state current and the transient current in an *RLC* circuit with $L = 4$ henries, $R = 8$ ohms, $C = \frac{1}{8}$ farad, and $E(t) = 16 \sin t$ volts. You may assume that $I(0) = 0$ and $I'(0) = 0$
23. Work Problem 22 for the case where $E(t) = 10te^{-t}$.

C 24. **The principle of superposition** Let y_1 be a solution to the second-order linear differential equation

$$y'' + P(x)y' + Q(x)y = F_1(x)$$

and let y_2 satisfy

$$y'' + P(x)y' + Q(x)y = F_2(x)$$

Show that $y_1 + y_2$ satisfies the differential equation

$$y'' + P(x)y' + Q(x)y = F_1(x) + F_2(x)$$

CHAPTER 14 REVIEW

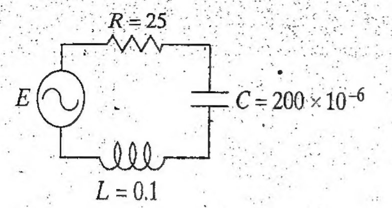
Proficiency Examination

CONCEPT PROBLEMS

1. What is a separable differential equation?
2. What is a homogeneous differential equation?
3. What is the form of a first-order linear differential equation?
4. What is an exact differential equation?
5. Describe Euler's method.
6. Define what it means for a set of functions to be linearly independent.
7. What is the Wronskian, and how is it used to test for linear independence?
8. a. What is the characteristic equation of $ay'' + by' + cy = 0$?
   b. What is the general solution of a second-order homogeneous equation?
9. Describe the form of the general solution of a second-order nonhomogeneous equation.
10. Describe the method of undetermined coefficients.
11. Describe the method of variation of parameters.

PRACTICE PROBLEM

12. An *RLC* circuit has inductance $L = 0.1$ henry, resistance $R = 25$ ohms, and capacitance $C = 200$ microfarads (i.e., 200×10^{-6} farad). If there is a variable voltage source of $E(t) = 50 \cos 100t$ in the circuit, what is the current $I(t)$ at time t? Assume that when $t = 0$, there is no charge and no current flowing.



Supplementary Problems

1. Consider the **Clairaut equation**

$$y = xy' + f(y')$$

a. Differentiate both sides of this equation to obtain

$$x + f'(y')]y'' = 0$$

Because one of the two factors in the product on the left must be 0, we have two cases to consider. What is the solution if $y'' = 0$? This is the *general solution*.

b. What is the solution if

$$x + f'(y') = 0$$

(This solution is called the *singular solution*.) *Hint*: Use the parameterization $y' = t$.

c. Find the general and singular solutions for the Clairaut equation

$$y = xy' + \sqrt{4 + (y')^2}$$

2. A differential equation of the form

$$x^2y'' + Axy' + By = F(x)$$

with $x \neq 0$ is called an **Euler equation**. To find a general solution of the related homogeneous equation

$$x^2y'' + Axy' + By = 0$$

we assume that there are solutions of the form $y = x^m$.

a. Show that m must satisfy

$$m^2 + (A - 1)m + B = 0$$

This is the characteristic equation for the Euler equation.

b. Distinct real roots. Characterize the solutions of the related homogeneous equation in the case where

$$(A - 1)^2 > 4B$$

c. Repeated real roots. Characterize the solutions of the related homogeneous equation in the case where

$$(A - 1)^2 = 4B$$

d. Complex conjugate roots. Suppose that

$$(A - 1)^2 < 4B$$

and that $\alpha \pm \beta i$ are the roots of the characteristic equation. Verify that $y_1 = x^\alpha \cos(\beta \ln|x|)$ is one solution. Use reduction of order to find a second solution y_2, then characterize the general solution.

3. Find the general solution of the Euler equation

$$x^2y'' + 7xy' + 9y = \sqrt{x}$$

Hint: Use the result of Problem 2 along with variation of parameters.

4. In certain biological studies, it is important to analyze *predator-prey* relationships. Suppose $x(t)$ is the prey population and $y(t)$ is the predator population at time t. Then these populations change at rates

$$\frac{dx}{dt} = a_{11}x - a_{12}y \qquad \frac{dy}{dt} = a_{21}x - a_{22}y$$

where a_{11} is the natural growth rate of the prey, a_{12} is the predation rate, a_{21} measures the food supply of the predators, and a_{22} is the death rate of the predators. Outline a procedure for solving this system.

5. Consider the almost homogeneous differential equation

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{rx + sy + t}\right)$$

with $as = br$ (recall Problem 36, Section 14.1).

a. Let $u = \dfrac{ax + by}{a}$. Show that $u = \dfrac{rx + sy}{r}$ and

$$\frac{du}{dx} = \frac{a}{b}\left(\frac{du}{dx} - 1\right).$$

b. Verify that by making the change of variable suggested in part **a**, you can rewrite the given differential equation in the separable form

$$\frac{du}{1 + \frac{b}{a} f\left(\frac{au + c}{ru + t}\right)} = dx$$

c. Use the procedure outlined in parts **a** and **b** to solve the almost homogeneous differential equation

$$\frac{dy}{dx} = \frac{2x + y - 3}{4x + 2y + 5}$$

6. **Putnam Examination Problem** Find all solutions of the equation

$$yy'' - 2(y')^2 = 0$$

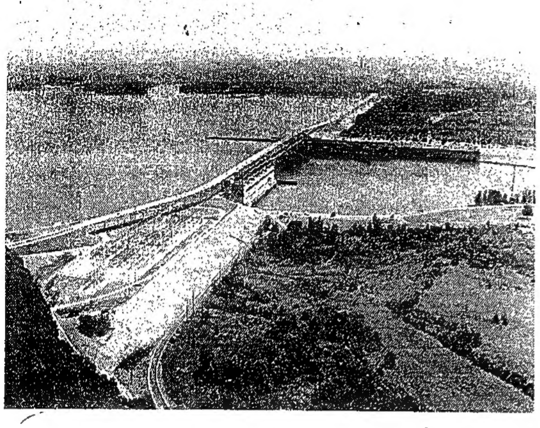
that pass through the point $x = 1, y = 1$.

7. **Putnam Examination Problem** A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If air resistance is neglected and the muzzle velocity is constant ($v(0) = v_0$), determine the set H of points in the plane and above the horizontal that can be hit.

8. **Putnam Examination Problem** Show that

$$x + \frac{2}{3}x^3 + \frac{2 \cdot 4}{3 \cdot 5}x^5 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}x^7 + \cdots = \frac{\sin^{-1} x}{\sqrt{1 - x^2}} + \frac{x}{2}$$

Save the Perch Project



This project is to be done in groups of three or four students. Each group will submit a single written report.

Happy Valley Pond is currently populated by yellow perch. A map is shown in Figure 14.12. Water flows into the pond from two springs and evaporates from the pond, as shown by the following table.

Spring	Dry Season	Rainy Season
A	50 gal/h	60 gal/h
B	60 gal/h	75 gal/h
Evaporation:	110 gal/h	75 gal/h

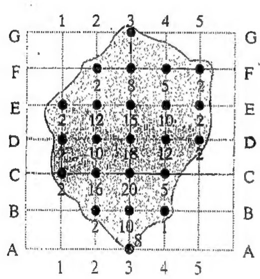


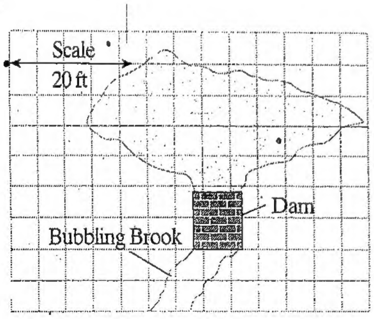
Figure 14.12 Happy Valley Pond is fed by two springs A and B

Unfortunately, spring B has become contaminated with salt and is now 10% salt, which means that 10% of a gallon of water from spring B is salt. The yellow perch will start to die if the concentration of salt in the pond rises to 1%. Assume that the salt will not evaporate but will mix thoroughly with the water in the pond. There was no salt in the pond before the contamination of spring B. Your group has been called upon by the Happy Valley Bureau of Fisheries to try to save the perch.

Your paper should include the number of gallons of water in the pond when the water level is exactly even with the top of the spillover dam. The overflow goes over the dam into Bubbling Brook. The following table gives a series of measurements of the depth of the pond at the indicated points when the water level was exactly even with the top of the spillover dam.

DEPTH OF HAPPY VALLEY POND (Location/depth)

A3/8 ft	B2/2 ft	B3/10 ft	B4/1 ft	C1/2 ft	C2/16 ft	C3/20 ft
C4/5 ft	D1/2 ft	D2/10 ft	D3/18 ft	D4/12 ft	D5/2 ft	E1/2 ft
E2/12 ft	E3/15 ft	E4/10 ft	E5/2 ft	F2/2 ft	F3/8 ft	F4/5 ft
F5/2 ft	G3/1 ft					



AMONG all the mathematical disciplines the theory of differential equations is the most important .... It furnishes the explanation of all those elementary manifestations of nature which involve time ....

—Sophus Lie
Leipziger Berichte, 47 (1895).

Let $t = 0$ hours correspond to the time when spring B became contaminated. Assume it is the dry season and that at time $t = 0$ the water level of the pond was exactly even with the top of the spillover dam. Write a differential equation for the amount of salt in the pond after t hours. Draw a graph of the amount of salt in the pond versus time for the next 3 mo. How much salt will there be in the pond in the long run, and do the fish die? If so, when do they start to die? It is very difficult to find where the contamination of spring B originates, so the Happy Valley Bureau of Fisheries proposed to flush the pond by running 100 gal of pure water per hour through the pond. Your report should include an analysis of this plan and any modifications or improvements that could help save the perch.

*This group project is courtesy of Diane Schwartz from Ithaca College, New York.